Junior Mathematical Olympiad 2014

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working or false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

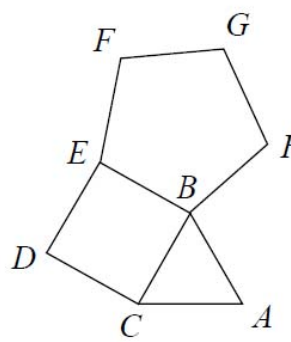
We have collected below a small number of solutions submitted for the JMO in 2014, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

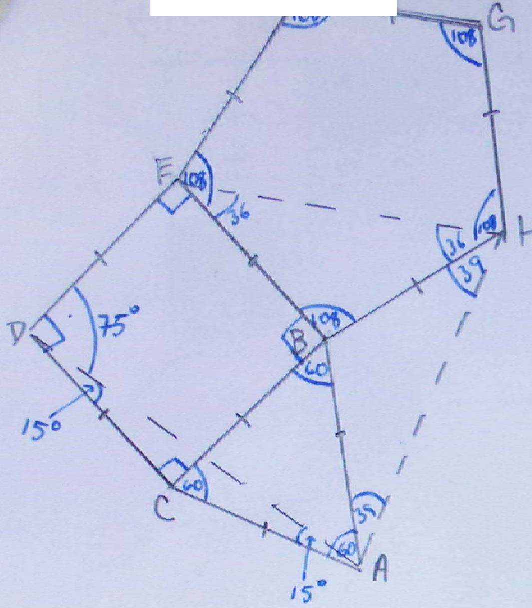
- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1. The figure shows an equilateral triangle ABC, a square $BCDE$, and a regular pentagon $BEFGH$.
What is the difference between the sizes of $\angle ADE$ and $\angle AHE$?



Solution



Not to Scale

All of the sides are of equal length in ΔABC, square BCDE and Pentagon BEFGH as they are all regular polygons with touching sides.

The angles in the pentagon $= \frac{3\times180}{5} = 108°$

The angles in the square $= 90°$

The angles in $\Delta ABC = 60°$

The line AD forms the ~~isosceles~~ isosceles Δ ADC

$\angle ACD = 90° + 60° = \underline{150°}$

Thus, $\angle CAD = \angle CDA = \frac{180-150}{2} = \underline{15°}$

$\angle CDE = 90°$ and $\angle CDA = 15°$, so $\angle ADE = 90° - 15° = \underline{75°}$

The line AH forms the isosceles triangle ABH in which AB = BH.

$\angle ABH = 360° - 60° - 90° - 108° = \underline{102°}$ (as angles around a point = 360°)

Thus, $\angle BAH = \angle BHA = \frac{180° - 102°}{2} = \underline{39°}$

The line HE forms the isosceles triangle BHE in which BH = BE

$\angle HBE = 108°$

Thus, $\angle BHE = \angle BEH = \frac{180-108}{2} = \underline{36°}$

$\angle AHB = 39°$ and $\angle BHE = 36°$ so $\angle AHE = 39+36 = \underline{75}$.

$\angle ADE = 75°$ and $\angle AHE = 75°$ so $\angle ADE = \angle AHE$

There is no difference.

B2. I start at the square marked A and make a succession of moves to the square marked B. Each move may only be made downward or to the right. I take the sum of all the numbers in my path and add 5 for every black square I pass through.

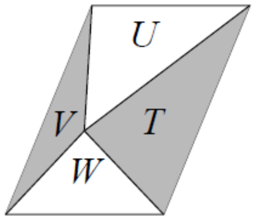
How many paths give a sum of 51?

A		12		10
	11		11	
10		10		15
	11		14	
10		13		B

Solution

B2 You ~~alwo~~ always pass through 4 black squares. $4 \times 5 = 20$ so the sum of the white squares must be $51 - 20 = 31$. You also always pass through 3 white squares. As they are all 2 digits, you need to pass through 10, 10 and 11. To get to B, you must pass through the diagonal line to the top left of it - 13, 14, 15. These stop the total of 51 from being possible, so there are no paths with this sum.

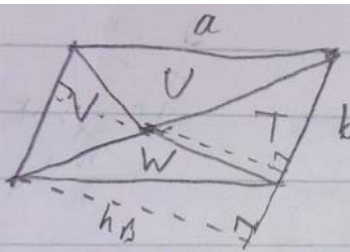
B3. A point lying somewhere inside a parallelogram is joined to the four vertices, thus creating four triangles T, U, V and W, as shown.



Prove that

$$\text{area } T + \text{area } V = \text{area } U + \text{area } W.$$

Solution



Let the sides of the parallelogram be a, b. Let h_T, h_U, h_V, h_W be the heights of T, U, V, W. Let h_B be the perpendicular distance in the diagram. The area of the parallelogram is bh_B.

$\therefore T + V = \frac{1}{2} bh_T + \frac{1}{2} bh_V = \frac{1}{2} b(h_T + h_V) = \frac{1}{2} bh_B$

$\therefore$ T + V is half the area of the parallelogram.

$\therefore$ area T + area V = area U + area W

as required

B4. There are 20 sweets on the table. Two players take turns to eat as many sweets as they choose, but they must eat at least one, and never more than half of what remains. The loser is the player who has no valid move.

Is it possible for one of the two players to force the other to lose? If so, how?

Solution

Start by taking 5 sweets. However many they take, you will be able to leave them 7 left: If they take 1, you take 7; if they take 2, you take 6; if they take 3, you take 5 and so on. When they have 7 left, they cannot take any number without letting you leave them 3: if they take 1, you take 3; if they take 2, you take 2 and if they take 3, you take 1. Once they have 3, they have no choice but to take 1. Then once you take 1, there is only 1 left, which they cannot take.

B5. Find a fraction $\frac{m}{n}$, with m not equal to n, such that all of the fractions

$$\frac{m}{n}, \frac{m+1}{n+1}, \frac{m+2}{n+2}, \frac{m+3}{n+3}, \frac{m+4}{n+4}, \frac{m+5}{n+5}$$

can be simplified by cancelling.

Solution

If m and n are multiples of 2 then m, n, m+2, n+2, m+4 and n+4 are all even so can all be cancelled down by at least 2.
If m and n are multiples of 3, m+3 and n+3 will both be multiples of 3 so can be cancelled by 3.
If m and n are multiples of 5, m+5 and n+5 will be multiples of 5, so can be cancelled down by 5.
So m and n (so far) are multiples of 2, 3 and 5. $2 \times 3 \times 5 = 30$ so they should both be multiples of 30.
All that remains is $m+1/n+1$.
~~Thus~~ Any multiple of 30 is divisible by 2, 3 and 5. Therefore such a number +1 will not be divisible by 2, 3 or 5. The next prime number is 7, so I am looking for 2 ~~numbers~~ multiples of 30 which are divisible by 7 when you add 1.

Multiples of 30	+1	divisible by 7?
30	31	No
60	61	No
90	91	$91 \div 7 = 13$
120	121	No
150	151	No
180	181	No
210	211	No
240	241	No
270	271	No
300	301	$301 \div 7 = 43$

I have now found my two numbers — 90 and 300.
To check:
$\frac{90}{300} = 3/10$
$91/301 = 13/43$
$92/302 = 46/151$
$93/303 = 31/101$
$94/$ … $= 47/$ …

So the values $m = 90$ and $n = 300$ suit this condition.

B6. The sum of four different prime numbers is a prime number. The sum of some pair of the numbers is a prime number, as is the sum of some triple of the numbers. What is the smallest possible sum of the four prime numbers?

Solution

There must be a 2 in the solution, otherwise the four numbers would add up to an even number which cannot be prime (with the exception of 2, which cannot be made by adding together four positive integers) The other three must all add up to get a prime number because two odd numbers add up to make an even number and so do two even numbers. The total must be greater than 17 because that is what the four lowest prime numbers add up to.

One pair of numbers which makes a prime is 2+3 which makes 5.

Prime numbers which can be made by adding together 3 primes are greater than 15 because that is what the three lowest odd primes add up to. It cannot be 17 because 3+5+7 = 15 and 3+5+11 = 19 (the second lowest possible combination) 19+2 is 21 so that is incorrect. 23+2 = 25 so that is also wrong. 29+2 = 31 that is right! The four primes are 2, 3, 7 and 19 which add together to make 31. 2+3 = 5. 3+7+19 = 29. 5, 29 and 31 are prime, therefore 31 is correct!