Junior Mathematical Olympiad 2013

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage them in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working or false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2013, with the aim that future candidates can see what some students in this age group do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared – for example, in B2 below, a good way to start the solution is by stating “Let Pippa’s number be p and Ben’s number be b”;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1 How many numbers less than 2013 are both:

(i) the sum of two consecutive positive integers; **and**

(ii) the sum of five consecutive positive integers?

Solution

Let's call any number which fulfills condition (i), "a". a must equal $2x+1$, where x is a natural number, since it is the sum of 2 consecutive natural numbers, i.e. $x+(x+1)$.

We can do something similar for condition (ii). Let's call any number which meets the criterion "b". This can be called $5x+10$:

$x+(x+1)+(x+2)+(x+3)+(x+4) = 5x+10$

Any number which meets both requirements is therefore both an odd number larger than 3 (due to $2x+1$), and a multiple of 5 (due to $5x+10$, which is the same as $5(x+2)$).

There are $(2010 \div 5)$ multiples of 5 below 2013, and half of them are odd. So we get 201 odd multiples of 5 below 2013. However, 5 does not meet condition (ii), and so there are 200 numbers which have these properties.

Answer: 200

B2 Pippa thinks of a number. She adds 1 to it to get a second number. She then adds 2 to the second number to get a third number, adds 3 to the third to get a fourth, and finally adds 4 to the fourth to get a fifth number.

Pippa's brother Ben also thinks of a number but he subtracts 1 to get a second. He then subtracts 2 from the second to get a third, and so on until he too has five numbers.

They discover that the sum of Pippa's five numbers is the same as the sum of Ben's five numbers. What is the difference between the two numbers of which they first thought ?

Solution

If we think of pippas number as x then we can say that her numbers are as follows:

x
$x+1$
$x+3$
$x+6$
$x+10$

So all her numbers together make $5x+20$

If We then think of Bens number as y then his numbers are:

y
$y-1$
$y-3$
$y-6$
$y-10$

So all his numbers together make $5y-20$.

We know that ~~all th~~ the total of Pippas numbers equals the sum of Bens numbers so

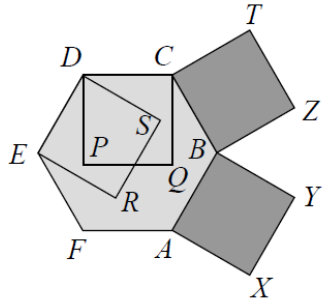
$5x+20 = 5y-20$

$5x+40 = 5y$

$x+8 = y$

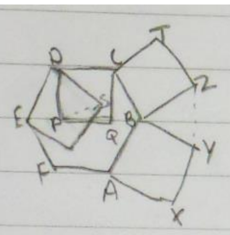
So the difference between their original numbers was 8.

B3 Two squares *BAXY* and *CBZT* are drawn on the outside of a regular hexagon *ABCDEF*, and two squares *CDPQ* and *DERS* are drawn on the inside, as shown.



Prove that $PS = YZ$.

Solution



As the squares share the same side as the regular hexag all the sides are equal. One interior angle of a regular hexag is $(6-2)\times180 = 4\times180 = 720$. $720 \div 6 = 120°$ = one interior angle a regular hexagon

As squares DCQP and DSRE overlap, the angle which they overl can be worked out ($\angle SDP$).

$$(90+90)-120 = \angle SDP$$
$$180-120 = 60 = \angle SDP$$

DS and PD are the same and the angle between them is 60° this means that $\triangle DSP$ is an equalateral.

$\angle CBA = 120°$ so the exterior angle of corner B, $360-120=2$

As there are two squares we can work out $\angle ZBY$.

$$240-(90+90) = \angle ZBY$$
$$240-180 = \angle ZBY$$
$$60° = \angle ZBY$$

ZB and YB are the same, and the angle inbetween them is 60°. This makes $\triangle ZYB$ equalateral. Also as all the lines are the same length the two equalateral triangles are the same, makin all the sides on the triangles the same. Therefore $PS = ZY$.

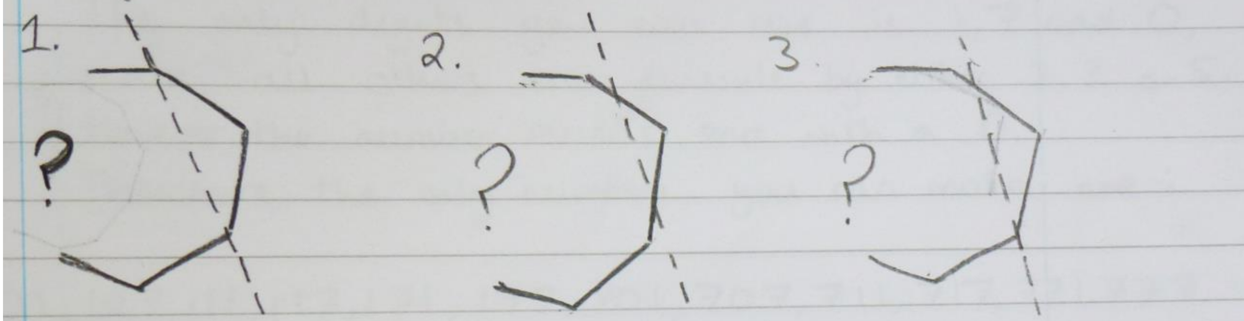
B4 A regular polygon P with n sides is divided into two pieces by a single straight cut. One piece is a triangle T, the other is a polygon Q with m sides.

How are m and n related?

Solutions

You could only make a triangle in 3 different ways:

1. 2. 3.



Bisecting through 2 angles, ↓ m is 1 less than n.

Bisecting through 2 sides ↓ m is 1 more than n.

Bisecting through 1 side and 1 angle. ↓ m is equal to n.

and

Since a triangle has three ~~two~~ sides, when you are cutting from corner to corner, you are removing two sides and creating 1 Therefore, ~~m th~~ m could be n-1

Another way of cutting would be from corner to side, which would mean removing one side, shortening yet preserving one side and adding one side. Therefore, m could be n

Another way of cutting would be from side to side, which would entitle shortening, yet preserving two sides and adding one. Therefore, m could be n+1.

Overall, there are three different relationships m could have with n. M could be n-1, n or n+1.

B5 Consider three-digit integers N with the two properties:

(a) N is not exactly divisible by 2, 3 or 5;

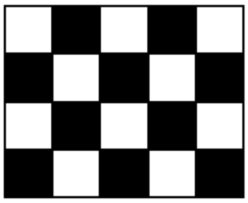
(b) no digit of N is exactly divisible by 2, 3 or 5.

How many such integers N are there?

Solution

There are no 3-digit numbers that have the properties that are described. You can Prove this by first taking out all of the numbers divisible by 2, 3 or 5. The list is: 0, 2, 3, 4, 5, 6, 8 and 9. This only leaves the numbers 1 and 7 avaliable but you cannot make any numbers with the properties of 'N' because 1 and 7 give remainders of 1 when divided by 3. This means that for 3 digit numbers you have 3 numbers with remainders of 1, they add to make 3. this shows that whatever number that is 3-digits long with 1 and 7 you cannot make a number with all the properties of 'N'.

B6 On the 4 × 5 grid shown, I am only allowed to move from one square to a neighbouring square by crossing an edge. So the squares I visit alternate between black and white. I have to start on a black square and visit each black square exactly once. What is the smallest number of white squares that I have to visit? Prove that your answer is indeed the smallest.



(If I visit a white square more than once, I only count it as one white square visited).

Solution

There are 10 squares in total.

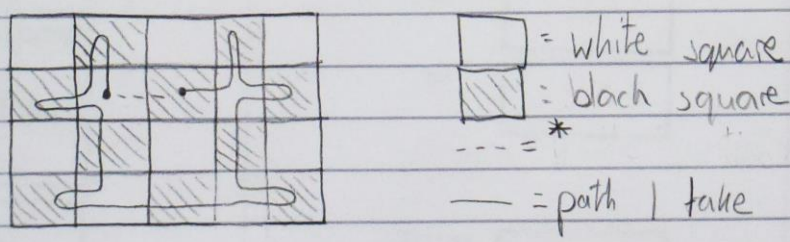
If 3 white squares were visited, let us call them a, b and c. 2 black squares would have to connect to two of them in order for one to be able to get from one white square to the next. Therefore the most black squares one could reach would be ~~then~~ 3+3+4 = 10.

But in this case, they would all have to be touching 4 black squares, and therefore they cannot be at an edge.

But if they are not at an edge, they cannot reach the ~~corner~~ black squares in the corner.

Therefore there cannot be only 3 white squares visited.

I have found a solution with 4 white squares, reaching each black square once and only once:



As there is a solution with 4 yet not with 3, the fewest white squares being visited while every black square is visited once and only once is 4.

* If the question means that I need to visit the first black square again.