Junior Mathematical Olympiad 2012

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working and false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2012, with the aim that future candidates can see what some Year 8 students (and younger ones) do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is, perhaps, not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain all subsequent details – however, the use of just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared – for example, in B2 below, a good way to start the solution is by stating "Let Anastasia's number be x";
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Out of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1 There was an old woman who lived in a shoe. She had 9 children at regular intervals of 15 months. The oldest is now six times as old as the youngest. How old is the youngest child?

Solutions

Answer: 2 years (24 months)

Say that the age of the youngest child is x, then the age of the oldest is $6x$. The difference between their ages is 120 months, so $6x - x = 120$ which means $5x = 126$, so $x = 24$. So the youngest child must be 24 months or 2 years old

and

Let the age of the youngest child be x, then the age of the eldest child must be $x + 10$, because there are 8 age gaps of 15 months between them, which means 8 years, plus another 2, for the 8 lots of extra 3 months. We also know that the eldest child's age is $6x$. so if $6x = x + 10$, we can solve the equation to find that $x = 2$. Therfore the youngest child's age must be 2.

B2 Anastasia thinks of a positive integer, which Barry then doubles. Next, Charlie trebles Barry's number. Finally, Damion multiplies Charlie's number by six. Eve notices that the sum of these four numbers is a perfect square. What is the smallest number that Anastasia could have thought of?

Solutions

Let us call Anastasia's number $= x$.
$\therefore$ Barry's number is $= 2x$.
Charlie's is $= 6x$.
and Damion's is $= 36x$.

$\therefore$ The total is $45x$.

Now, $45 = 3^2 \times 5$.
$\therefore$ The smallest x can be to make $45x$ square is 5, which makes
$45x = 225 = 15^2$.

$\therefore x = 5$.

and

B2) Let Anastasia's number $= x$
Barry doubles it : $2x$
Charlie trebles it : $6x$
Damion multiplies it by 6 : $36x$
Eve says the sum of the numbers equals a perfect square: $x + 2x + 6x + 36x = 45x$
If x was 1, the answer would be 45, which is not a square number
If x was 2, the answer would be 90, which is not a square number
If x was 3, the answer would be 135, which is not a square number
If x was 4, the answer would be 180, which is not a square number
If x was 5, the answer would be 225 which is a perfect square.
Therefore, the smallest number x could be is five.

B3 Mr Gallop has two stables which each initially housed three ponies. His prize pony, Rein Beau, is worth £250 000. Usually Rein Beau spends his day in the small stable, but when he wandered across into the large stable, Mr Gallop was surprised to find that the average value of the ponies in each stable rose by £10 000. What is the total value of all six ponies?

Solution

Let the total value of the ponies in the small stable be x, and in the large stable be y.

Then, from the information given:

$$\frac{x-250{,}000}{2} = \frac{x}{3} + 10{,}000$$

$$3x - 750{,}000 = 2x + 60{,}000$$

$$x = 810{,}000$$

Now looking at the large stable:

$$\frac{y+250{,}000}{4} = \frac{y}{3} + 10{,}000$$

$$3y + 750{,}000 = 4y + 120{,}000$$

$$y = \cancel{738{,}000}\ 630{,}000$$

$\therefore$ The total value $= x + y = 630{,}000 + 810{,}000$

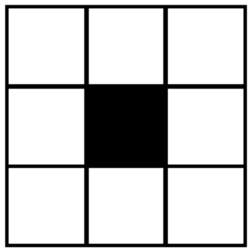
$= \cancel{£144{,}000}$

$= £1{,}440{,}000$

B4 An irregular pentagon has five different interior angles each of which measures an integer number of degrees. One angle is 76°.

The other four angles are three-digit integers which fit one digit per cell across and down into the grid on the right.

In how many different ways can the grid be completed?



Solution

A	B	C
D		E
F	G	H

the angles in a pentagon add up to ~~3~~ 540°

~~44~~ 540° − 76° = 464° ∴ the 4 unknown angles add up to 464°

~~all the~~ no number can be less than 100 ∴ the highest number possible is 164

∴ A, C & F are all 1

1	B	1
D		E
1	G	H

the units must add up to 4, ~~[illegible]~~

∴ H must be 1 or 6

if H is 1, then the numbers are 1B1, 1D1, 1E1 & 1G1

464 − 404 = 60 ∴ the 10s ~~columns~~ digits must add up to 6

all the numbers must be different, therefore they must be 0, 1, 2 & 3.

there are 24 ways of arranging these numbers between B, D, E and G

if H is 6, then the numbers are 1B1, 1D1, 1E6, 1G6.

464 − 414 = 50 ∴ the 10s digits must add up to 5.

this can be done with ~~[illegible]~~, 0023, 0014, 0113, 0122

B and D cannot be the same and E and G cannot be the same

the possibilities are in the table below

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
B	0	0	0	0	0	0	0	0	0	0	0	0	1	1	4	4	2	2	3	3	1	1	2	2	3	1	1	2
D	2	2	3	3	1	1	4	4	1	1	2	2	0	0	0	0	0	0	0	0	0	0	0	0	1	3	2	1
E	0	3	2	0	0	4	1	0	1	3	1	2	0	4	1	0	0	3	2	0	1	3	1	2	1	1	2	2
G	3	0	0	2	4	0	0	1	3	1	2	1	4	0	0	1	3	0	0	2	3	1	2	1	0	0	0	0

29	30	31	32
3	1	1	2
1	3	2	1
0	0	0	0
1	1	2	[illegible]

as shown there are 32 possibilities

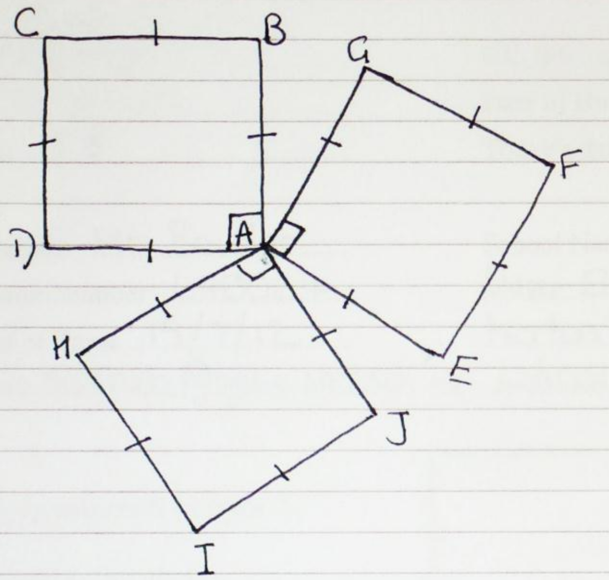
24 + 32 = 56

B5 Three identical, non-overlapping, squares *ABCD*, *AEFG*, *AHIJ* (all labelled anticlockwise) are joined at the point *A*, and are 'equally spread' (so that $\angle JAB = \angle DAE = \angle GAH$). Calculate $\angle GBH$.

Solution

There were two possible configurations for the diagram in this question. Here is a solution for each configuration:

From the information given, the squares would look like this:



Angles about a point add up to 360° so:

$\angle BAG = \angle EAJ = \angle DAH = 30°$ (The three angles are identical if $\angle JAB = \angle DAE = \angle GAH$)

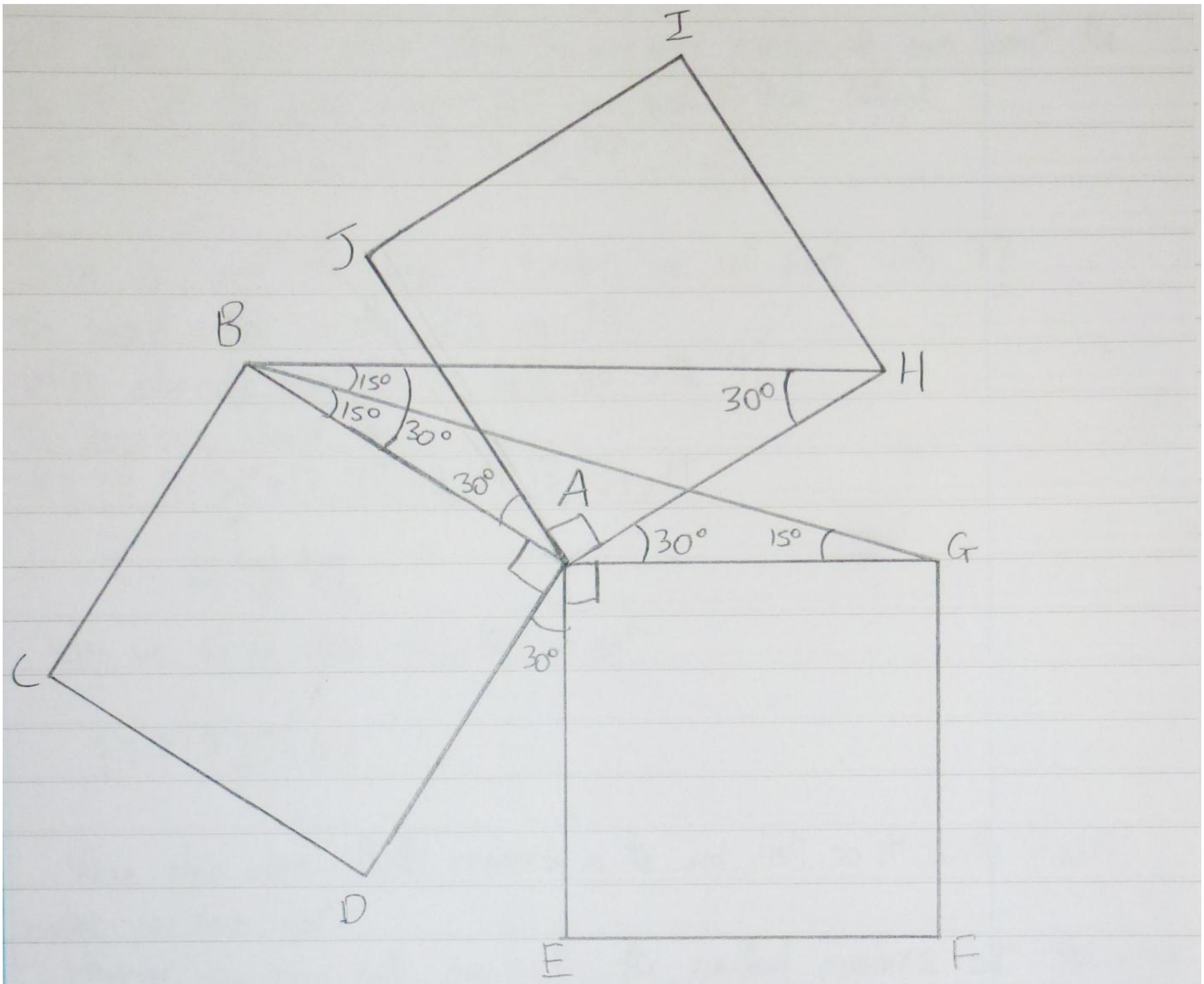
We can now calculate $\angle GBA$. ΔABG is isosceles so:

$$\angle ABG = \frac{180° - 30°}{2} = 75°$$

We can also calculate $\angle ABH$. ΔABH is also isosceles so:

$$\angle ABH = \frac{180° - 120°}{2} = 30°$$

This means that $\angle GBH = 75° + 30° = 105°$



Since $\angle JAB = \angle DAE = \angle GAH$, the corners of the squares are right angles, and they all form a full 360°, they are all:

$$\frac{360-90\times 3}{3} = \frac{360-270}{3}$$
$$=120-90$$
$$=30°$$

We can also work out that $\angle ABG$ is $\frac{180-\overset{(\angle GAB)}{(90+30+30)}}{2} = 15°$ because ΔBAG is iscoceles and the angles in a triangle add up to 180° rule.

From the same observations we can work out that $\angle ABH$ is $\frac{180-\overset{(\angle HAB)}{(90+30)}}{2} = 30$

$\angle ABH - \angle ABG = \angle GBH = 30-15 = 15°$

$\angle GBH = 15°$

B6 The integer 23173 is such that

(a) every pair of neighbouring digits, taken in order, forms a prime number;

and (b) all of these prime numbers are different.

What is the largest integer which meets these conditions?

Solution

6 The prime numbers less than 100 are: 02, 03, 05, 07, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

All those beginning with 0, 2, 4, 5, 6 or 8 can only be used at the beginning, so I'll leave them for later, when I can put them at the beginning.

The remaining primes are:

First digits

4 1's: 11, 13, 17, 19

2 3's: 31, 37

3 7's: 71, 73, 79

1 9: 97

3 1's, 2 3's 3 7's, and 2 9's

Last digits

In order to include all the primes, the number must have last digit 9 and second digit 1.

We can make the biggest number by using the biggest prime possible at each step.

We have:

61 19 97 73 37 71 13 31 11 17 79

second digit must be 1

cannot end in 9, as I already used

17 would force 79, which ends in 9

only numbers left are 17 and 79, so use

So the largest integer which meets those conditions is 6197371311179.