UK Junior Mathematical Olympiad 2016

Organised by The United Kingdom Mathematics Trust

Tuesday 14th June 2016

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. **Write in blue or black pen or pencil.**
   For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.
   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.
   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.
   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 30 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you are not able to do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

Section A

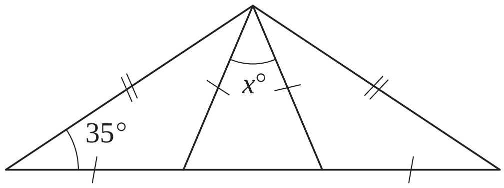
Try to complete Section A within 30 minutes or so. Only answers are required.

A1. Roger picks two consecutive integers, one of which ends in a 5. He multiplies the integers together and then squares the result.

What are the last two digits of his answer?

A2. Three isosceles triangles are put together to create a larger isosceles triangle, as shown.

What is the value of x?

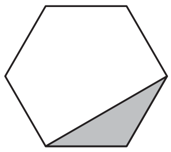


A3. The first term of a sequence is 0. Each term of the sequence after the first term is equal to $10p + 1$, where p is the previous term.

What is the sum of the first ten terms?

A4. The diagram shows a regular hexagon with area $48\ \text{m}^2$.

What is the area of the shaded triangle?



A5. Linda has a very thin sheet of paper measuring 20 cm by 30 cm. She repeatedly folds her paper in half by folding along the shorter line of symmetry. She finishes when she has a rectangle with area $75\ \text{cm}^2$.

What is the perimeter of her final rectangle?

A6. The points A, B, C, D and E lie in that order along a straight line so that $AB : BC = 1 : 2$, $BC : CD = 1 : 3$ and $CD : DE = 1 : 4$. What is $AB : BE$?

A7. A certain positive integer has exactly eight factors. Two of these factors are 15 and 21.

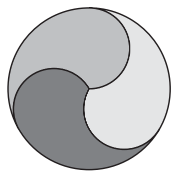
What is the sum of all eight factors?

A8. Julie and her daughters Megan and Zoey have the same birthday. Today, Julie is 32, Megan is 4 and Zoey is 1.

How old will Julie be when her age is the sum of the ages of Megan and Zoey?

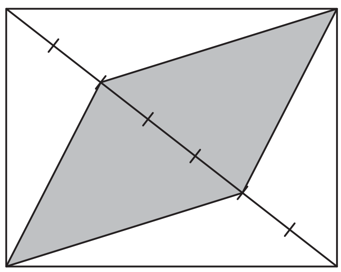
A9. A circle of radius 18 cm is divided into three identical regions by the three semicircles, as shown.

What is the length of the perimeter of one of these regions?



A10. The diagram shows a rectangle with length 9 cm and width 7 cm. One of the diagonals of the rectangle has been divided into seven equal parts.

What is the area of the shaded region?



Section B

Your solutions to Section B will have a major effect on your JMO result. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1. In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is $30°$ larger than the average of the other two angles.

What is the largest possible size of an angle in this triangle?

B2. The points A, B and C are the centres of three circles. Each circle touches the other two circles, and each centre lies outside the other two circles. The sides of the triangle ABC have lengths 13 cm, 16 cm and 20 cm.

What are the radii of the three circles?

B3. A large cube consists of a number of identical small cubes. The number of small cubes that touch four other small cubes face-to-face is 168.

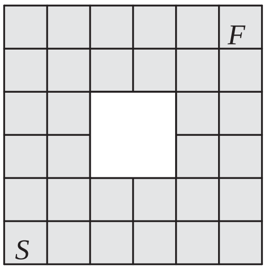
How many small cubes make up the large cube?

B4. In the trapezium $ABCD$, the lines AB and CD are parallel. Also $AB = 2DC$ and $DA = CB$.

The line DC is extended (beyond C) to the point E so that $EC = CB = BE$. The line DA is extended (beyond A) to the point F so that $AF = BA$.

Prove that $\angle FBC = 90°$.

B5. The board shown has 32 cells, one of which is labelled S and another F. The shortest path starting at S and finishing at F involves exactly nine other cells and ten moves, where each move goes from cell to cell 'horizontally' or 'vertically' across an edge.



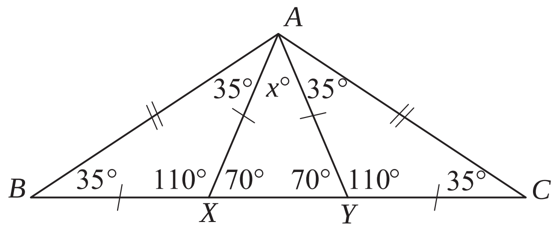
How many paths of this length are there from S to F?

B6. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

UK Junior Mathematical Olympiad 2016 Solutions

A1 00 Since the integers are consecutive and one ends in a 5, the other integer ends in a 4 or a 6 and so is even. Hence the product of the two numbers is a multiple of 10. When this is squared, we obtain a multiple of 100 and so the final digits are 00.

A2 40



Using the angle sum of a triangle, the isosceles triangles ABX and ACY have angles of 35°, 35° and $180° - 35° - 35° = 110°$. Angles on a straight line add to 180° and so $\angle AXY = \angle XYA = 180° - 110° = 70°$. Using the angle sum of a triangle again, we have $x = 180 - 70 - 70 = 40$.

Alternative Solution

Since $\triangle ABC$ is isosceles, $\angle ACB = 35°$.

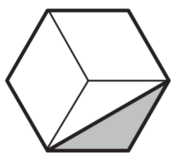
Since $\triangle ABX$ is isosceles, $\angle BAX = 35°$.

Since $\triangle ACY$ is isosceles, $\angle CAY = 35°$.

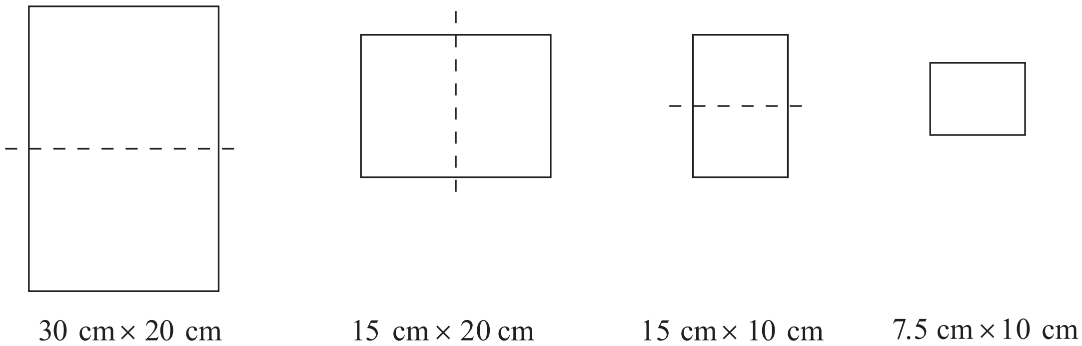
Hence, from the angle sum in $\triangle ABC$ we have $x = 180 - 4 \times 35 = 40$.

A3 123456789 The sequence is 0, 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111, … . Hence the sum of the first 10 terms is 123456789.

A4. 8 m^2 From the dissection shown, the area of the triangle is half of the area of a parallelogram which is itself a third of the area of the hexagon. So the triangle has area $\frac{1}{2} \times \frac{1}{3} \times 48\text{ m}^2 = 8\text{ m}^2$.



A5. 35 cm Each fold reduces the area by half. Therefore we require three folds to reduce the area from 600 cm^2 to 75 cm^2.



From the diagrams, we see that we have, in turn, areas of 600 cm^2, 300 cm^2, 150 cm^2 and 75 cm^2. The final shape has perimeter $2(7.5 + 10)\text{ cm} = 35\text{ cm}$.

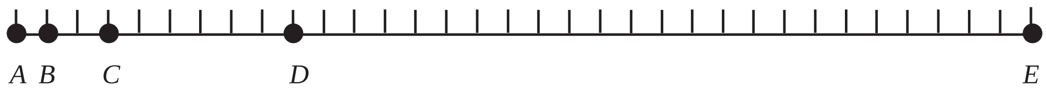
A6. 1:32 We have $AB : BC = 1 : 2$

and $BC : CD = 1 : 3 = 2 : 6$

and $CD : DE = 1 : 4 = 6 : 24$

and so $AB : BC : CD : DE = 1 : 2 : 6 : 24.$

Hence we have $AB : BE = 1 : (2 + 6 + 24) = 1 : 32$



A7. 192 We have $15 = 1 \times 3 \times 5$ and $21 = 1 \times 3 \times 7$. The integer we require is therefore a multiple of $1 \times 3 \times 5 \times 7 = 105$. However, 105 has the eight factors 1, 3, 5, 7, 15, 21, 35 and 105 and any (larger) multiple of 105 would have more than eight factors. Hence 105 is the integer we require and the sum of its factors is

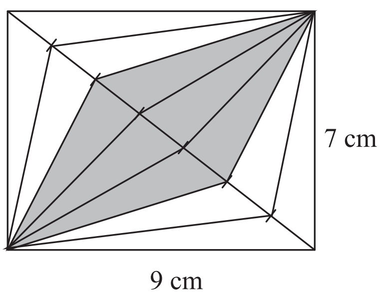
$$1 + 3 + 5 + 7 + 15 + 21 + 35 + 105 = 192.$$

A8. 59 Suppose this happens in x years' time. Julie's age will be $32 + x$, Megan's age will be $4 + x$ and Zoey's age will be $1 + x$. Thus $1 + x + 4 + x = 32 + x$ and, subtracting $5 + x$ from each side of this equation, we have $x = 27$.

In 27 years' time, Julie will be 59, Megan 31 and Zoey 28. Notice that $31 + 28 = 59$ as required.

A9. 30π Each region has a perimeter consisting of the perimeters of a third of a circle of radius 18 cm and two semicircles of radius 9 cm. Using the fact that the perimeter of a circle is π times its diameter, the required perimeter (in cm) is $\frac{1}{3} \times \pi \times 36 + 2 \times \frac{1}{2} \times \pi \times 18 = 30\pi$.

A10. 27 The 14 small triangles shown have equal heights and bases of equal length. Thus they have equal areas and their total area is that of the rectangle. The area of the rectangle is $7\,\text{cm} \times 9\,\text{cm} = 63\,\text{cm}^2$.



The area we require is made up of six small triangles.

Hence, the required area is $\frac{6}{14} \times 63\,\text{cm}^2 = 27\,\text{cm}^2$.

B1 In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is 30° larger than the average of the other two angles.

What is the largest possible size of an angle in this triangle?

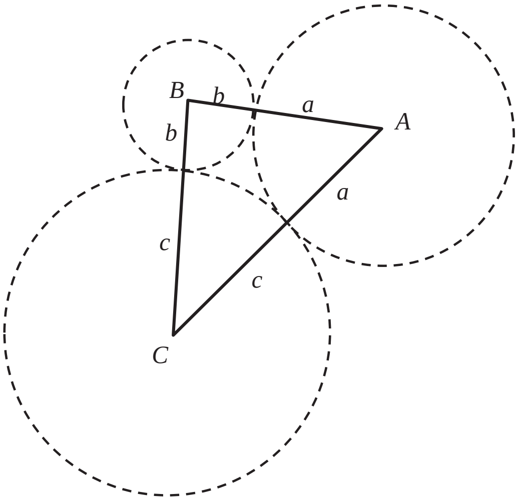
Solution

Let two of the angles measure $a°$ and $b°$. Then the third angle is $\left(30 + \frac{1}{2}(a + b)\right)°$. The angle sum of a triangle is 180° which gives $180° = \left(a + b + 30 + \frac{1}{2}(a + b)\right)°$ $= \left(30 + \frac{3}{2}(a + b)\right)°$ and so $\frac{3}{2}(a + b) = 150$, giving $a + b = 100$. The sizes of all the angles are integers so that the largest either a or b can be is 99. This gives a triangle with angles 1°, 80° and 99° and so the largest possible such angle is 99°.

B2 The points A, B and C are the centres of three circles. Each circle touches the other two circles, and each centre lies outside the other two circles. The sides of the triangle ABC have lengths 13 cm, 16 cm and 20 cm.

What are the radii of the three circles?

Solution



Since the circles touch, for each pair of circles, the distance between their centres is the sum of their radii. Let the radii of the three circles (in cm) be a, b and c. Then we can form equations for the lengths of the sides of the triangle:

$$a + b = 13$$

$$b + c = 16$$

$$a + c = 20.$$

Adding these equations together, we obtain

$$2a + 2b + 2c = 49.$$

So $a + b + c = 24\frac{1}{2}$. But $a + b = 13$ giving $c = 11\frac{1}{2}$. Then $b = 16 - 11\frac{1}{2} = 4\frac{1}{2}$ and $a = 20 - 11\frac{1}{2} = 8\frac{1}{2}$.

B3. A large cube is made up of a number of identical small cubes. The number of small cubes that touch four other small cubes face-to-face is 168.

How many small cubes make up the large cube?

Solution

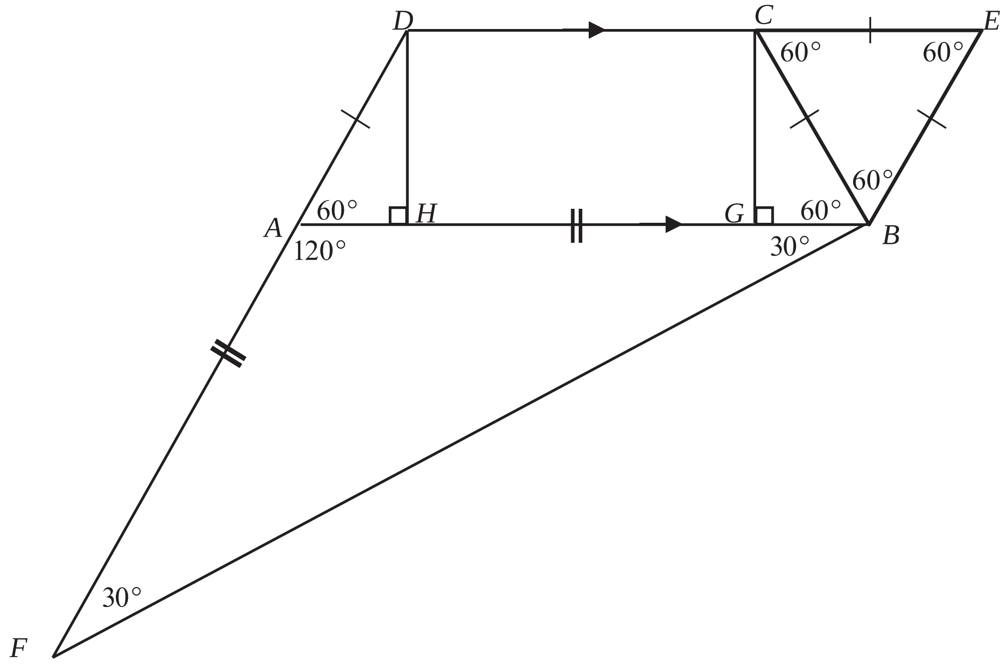
On a face of the large cube, the cubes not on an edge touch five other faces and the internal cubes not on a face touch six other faces. Small cubes which touch four other faces lie along an edge and are not at a corner (those touch 3 other faces). A cube has 12 edges. Now $\frac{168}{12} = 14$ so that means each edge of the large cube has 14 small cubes which touch four other cubes and two small corner cubes. Therefore an edge of the large cube has length 16 and so the total number of small cubes is $16 \times 16 \times 16 = 4096$.

Alternative Solution

The large cube has 8 corner cubes each of which has 3 faces which touch matching cubes. The cubes which lie on the edges between a pair of corner cubes each have 4 faces which touch matching cubes and we know there are 168 of these. A cube has 12 edges so dividing 168 by 12 gives 14 which tells us that each edge of the large cube has 16 small cubes in it. Thus the number of small cubes is $16 \times 16 \times 16 = 4096$.

B4. In the trapezium $ABCD$, the lines AB and CD are parallel. Also $AB = 2DC$ and $DA = CB$. The line DC is extended (beyond C) to the point E so that $EC = CB = BE$. The line DA is extended (beyond A) to the point F so that $AF = BA$.
Prove that $\angle FBC = 90°$.

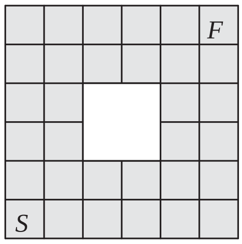
Solution



Since $EC = CB = BE$, the triangle ECB is equilateral and each of its angles is 60°.
ED and AB are parallel so $\angle ABC = \angle BCE = 60°$ (alternate angles).
Draw perpendiculars from C and D to AB to meet AB at G and H. The right-angled triangles DAH and CBG have hypotenuses of equal length and $DH = CG$ so triangles DAH and CGB are congruent {RHS}. Thus $\angle DAH = \angle CBG = 60°$.
DAF is a straight line so that $\angle BAF = 180° - 60° = 120°$.
Triangle AFB is isosceles with $\angle AFB = \angle FBA = \frac{1}{2}(180° - 120°) = 30°$ (angle sum of a triangle).
Therefore

$$\angle FBC = \angle FBA + \angle ABC = 30° + 60° = 90°.$$

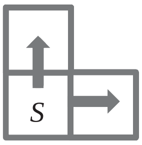
B5. The board shown has 32 cells, one of which is labelled *S* and another *F*. The shortest path starting at *S* and finishing at *F* involves exactly nine other cells and ten moves, where each move goes from cell to cell ‘horizontally’ or ‘vertically’ across an edge.



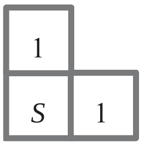
How many paths of this length are there from *S* to *F*?

Solution

From *S*, there are two starting paths:



and we can represent the number of paths to reach a square like this:



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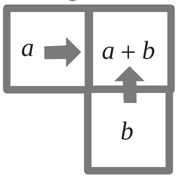
There is only one way to travel along the edges and we can build up our diagram like this, noticing that each square not on an edge can only be reached from the left or from below



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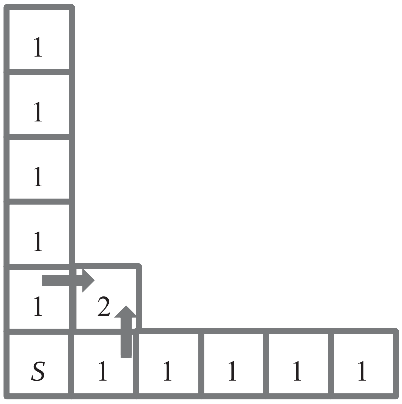

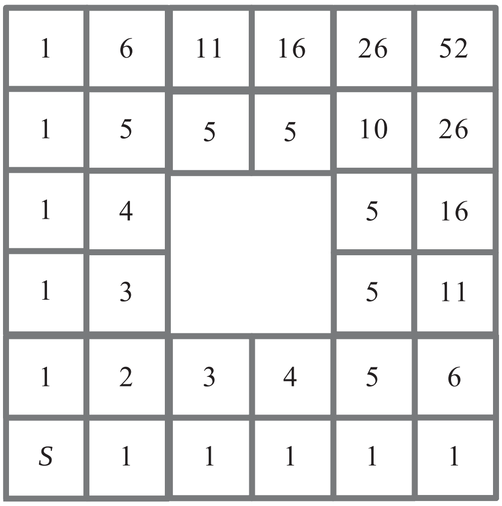

Since *F* is 5 steps above and 5 steps to the right of *S*, each of the 10 moves can only be upwards or to the right and so there are 52 paths from *S* to *F*.

B6. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

Solution

$n = 1$: it is impossible to place the integers 1, 2 and 3 into three groups with the same sum.

$n = 2$: $1 + 2 + 3 + 4 + 5 + 6 = 21$ and so we want to place these integers into three groups, each of which add to 7. The groups are (1, 6), (2, 5) and (3, 4).

$n = 3$: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ and so we want to place these integers into three groups, each of which adds to 15. Such a grouping is (1, 2, 3, 4, 5), (6, 9) and (7, 8).

It would not be sensible to continue to look at each value of n, so we look at what happens when we move from $3n$ to $3(n + 2)$.

Let us assume that we can place $3n$ integers into three groups with the same sum. For the first $3(n + 2)$ integers we must include an extra six integers $3n + 1$, $3n + 2$, $3n + 3$, $3n + 4$, $3n + 5$ and $3n + 6$ to deal with $3(n + 2) = 3n + 6$.

When we add the six new integers together, we obtain $18n + 21$ meaning we increase the sum by a third of this, $6n + 7$, to each group to make new groups with the same sum.

So we add pairs to our original three groups: $3n + 1$ with $3n + 6$ to one group; $3n + 2$ with $3n + 5$ to another group and $3n + 3$ with $3n + 4$ to the third group.

However, we know we can obtain three groups with the same sum for $n = 2$ and $n = 3$ and now we can also obtain three groups for each of $n = 4, 6, 8, \ldots$ and so every even integer, in the same way from $n = 3$ we have $n = 5, 7, 9, \ldots$ and so every odd integer greater than 1.

Thus the values of n for which the first $3n$ positive integers can be placed in three groups with the same sum are all values of $n > 1$.