Junior Mathematical Olympiad 2010

The Junior Mathematical Olympiad (JMO) has long aimed to help introduce able students to (and to encourage in) the art of problem-solving and proof. The problems are the product of the imaginations of a small number of volunteers writing for the JMO problems group. After the JMO, model solutions for each problem are published in the solutions booklet and the UKMT Yearbook.

However, these neatly printed solutions usually represent only one approach (often where several are feasible) and convey none of the sense of investigation, rough-working and false-starts that usually precede the finished article.

In marking JMO scripts, markers want to encourage complete (and concise) proofs in section B but, in doing so, are content to accept less than perfectly written answers, as long as it is clear where the proof is going – we appreciate that there is a limited time available, and that to polish and to present takes the place of considering another problem. Nevertheless, clarity and insight are what are looked for.

We have collected below a small number of solutions submitted for the JMO in 2010, with the aim that future candidates can see what some Year 8 students (and younger ones) do achieve and that they might aspire to emulate it. In many ways they are ordinary solutions, not brilliantly and startlingly original – but mathematically they are to be commended, in particular, for the logical progression from one point to the next, and for clear presentation.

It is not easy to generalise about what makes a good solution or how a candidate can achieve success in the JMO, but there are a few points to ponder on:

- using trial and improvement to a large degree is generally considered not very mathematical – it will lend little insight into the structure of the problem, and even if you get an answer that works, there should always be a concern that it is not the only answer;
- if you do have to resort to calculating your way through a large number of cases, then the calculations should be shown as part the solution – the reader should not have to make suppositions about what you have tried (or not tried);
- diagrams should be large enough to contain subsequent details – however, drawing just one diagram can convey very little of the order of the proof, especially in geometrical questions;
- avoid long sentences, particularly those which frequently use *if* and *could be*.
- the use of algebra makes it possible to express connections in simple ways, where a huge number of words would be otherwise necessary;
- if algebra is used, variables should be fully declared – for example, in B3 below, saying merely "Let Jack be x" led to much consternation for both candidates and markers over whether speed or time was being referred to;
- attempt as many questions as you can do really well in – it is better, in terms not only of scoring marks but also of honest satisfaction, to spend your time concentrating on a few questions and providing full, clear and accurate solutions, rather than to have a go at everything you can, and to achieve not very much in any of them. Of the 1000 students or so who take part in the JMO, generally only a handful achieve full marks in all six section B questions.

If this all sounds rather negative, it has, in balance, to be said that JMO candidates produce an astonishing amount of work that is worthy of high praise. We hope that the work below will give future candidates a flavour of what is to be encouraged, in style, method and presentation.

B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

If you write out the terms in terms of x and y, where x is the first term and y is the second, you get:

$$x,\ y,\ (x+y),\ (x+2y),\ (2x+3y),\ (3x+5y)$$

The last term is 4 times bigger than the first, so

$$4x = 3x + 5y$$

so $x = 5y$.

All of the terms add up to 13.

$$x + y + x + y + x + 2y + 2x + 3y + 3x + 5y = 13$$

$$8x + 12y = 13$$

We can substitute the x's for y as $x = 5y$.

So $40y + 12y = 13$

$52y = 13$

Therefore $y = 1/4$

$x = 5y$

so $x = 5/4$

To check this, these are the terms in order.

$$\tfrac{5}{4},\ \tfrac{1}{4},\ \tfrac{6}{4},\ \tfrac{7}{4},\ \tfrac{13}{4},\ \tfrac{20}{4}.$$

The last term is 4 times the first term. $\frac{5}{4} \times 4 = \frac{20}{4}$

The terms add up to 13. they add up to $\frac{52}{4}$ which is 13.

So the first term is $1\frac{1}{4}$.

B2 The eight-digit number "*ppppqqqq*", where p and q are digits, is a multiple of 45.

What are the possible values of p?

A number is a multiple of 45 if and only if it is divisible by both 5 and 9.

If a number is divisible by 5, its last digit must be either 0 or 5. So q is 0 or 5.

For a number to be divisible by 9, the sum of its digits must also be divisible by 9, so $4p+4q=4(p+q)$ is divisible by 9.

Now consider $4(p+q)$. As this is divisible by 9 and 4 shares no common factor with 9, $p+q$ must be divisible by 9. So if $q=0$, $p=9$ is the only possibility (as $p+q \neq 0$ and $p+q=18$ requires $p=18$ which is impossible) and if $q=5$, $p=4$ with the same reasoning as above.

Hence there are two possible values for p, $p=4$ and $p=9$.

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

How long did Jill actually take to walk up to the top of the hill?

Is $1.5 \times$ Jill's speed $= \frac{1}{2} \times$ Jack's speed, Jack was three times as fast as Jill. Therefore, Jill took three times as long as Jack.

Jill's time = Jack's time + Jack's time + Jack's time

Also, Jill's time minus Jack's time = $1\frac{1}{2}$ hours

Jack's time + Jack's time = $1\frac{1}{2}$ hours

Jack's time = $\frac{3}{4}$ hours

Jill's time = $3 \times$ Jack's time, so Jill's time = $\frac{9}{4}$ hours or $2\frac{1}{4}$ hours

Jill took 2 hours 15 minutes or $2\frac{1}{4}$ hours to walk up the hill.

Jack's speed = x

Jill's speed = y

50% of x = 150% of y

$$\frac{x}{2} = 1.5y$$

$$x = 3y$$

Therefore Jack was travelling 3 times faster than Jill and her time would be 3 times his.

Jack's time = g

$$g + 90 = 3g$$
$$90 = 2g$$
$$45 = g$$

Jacks time + 90 mins = Jills time ($3g$) so 90 mins = $2g$ and Jacks time = 45 mins

45×3 = jill's time

= 135 mins (2hrs 15mins)

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

Clues

Across

1. A triangular number
3. A triangular number

Down

1. A square number
2. A multiple of 5

1	2
3	

Firstly, I know that the bottom right-hand corner is either 5 or 0. Let's start with 5.

	5

3. Across is a triangular number, which ends in 5, so we have three choices: 15, 45 and 55.

1	5

4	5

5	5

Now we need a square number number that ends in 1, 4 or 5

8	
1	5

6	
4	5

2	
5	5

Finally, going across, a triangular number.

8	
1	5

6	6
4	5

2	1/8
5	5

Now we have 3 possibilities.

Now, on to 0.

	0

Only 1 choice now for a triangular number, that's 10.

1	0

Again, only 1 choice for a square number.

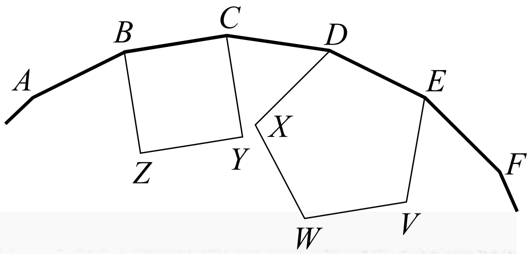
8	
1	0

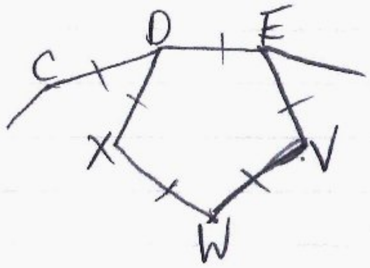
But this time there's no triangular number that has 8 tens.

So, how many different ways can the crossnumber be completed correctly? 3.

B5 The diagram shows part of a regular 20-sided polygon (an *icosagon*) *ABCDEF*…, a square *BCYZ* and a regular pentagon *DEVWX*.

Show that the vertex *X* lies on the line *DY*.





Exterior angles of a polygon add to 360°

$\therefore$ One exterior angle of the regular pentagon $= 360° \div 5$
$= 72°$ and

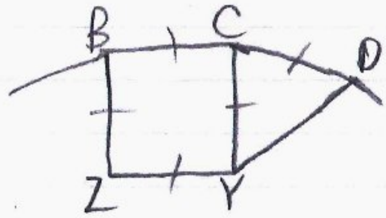
one exterior angle of the regular icosagon $= 360° \div 20$
$= 18°$.

$\therefore$ Interior angle of pentagon $= 180° - 72°$
$= 108°$ and

interior angle of icosagon $= 180° - 18°$
$= 162°$.

$\therefore \angle XDE = 108°$ and $\angle CDE = 162°$

$\therefore \angle CDX = 162° - 108°$
$= 54°$



$\angle BCD = 162°$ (angle of icosagon)

$\angle BCY = 90°$ (angle of square)

$\therefore \angle YCD = 162° - 90°$
$= 72°$

CD = BC (icosagon is regular) = CY (BCYZ is a square)

$\therefore \triangle CYD$ is isosceles

$\therefore \angle CDY = \angle CYD = (180° - 72°) \div 2$
$= 108° \div 2$
$= 54°$

$\therefore \angle CDX = \angle CDY = 54°$

$\therefore$ Because both angles are on the interior of the icosagon, X lies on the line DY. QED

B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total that the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars

I started by putting the jars in order so that the jars had more sweets in as you went from left to right:

$a < b < c < d < e$

and labeled the number of sweets a, b, c, d and e.

As all five numbers are whole numbers
$d \geqslant b+2$ and $e \geqslant c+2$
So $d+e \geqslant b+2+c+2$
From what I am told in the question
$a+b+c > d+e$

So $a+b+c > d+e \geqslant b+c+4$
This obviously means that $a > 4$
i.e. 5 or bigger

I tried to see what would happen if I put 5, 6, 7, 8 and 9 sweets in the jars, and saw that $5+6+7 > 8+9$

If the three smallest numbers 5, 6 & 7 have a bigger total than the two larger ones, 8 and 9, any other three numbers from 5, 6, 7, 8 & 9 will definitely have a larger total than the other two

So 5, 6, 7, 8 and 9 work, and therefore the smallest total number of sweets is
$5+6+7+8+9 = 35$