Junior Mathematical Challenge

Thursday 30th April 2015

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Junior Mathematical Challenge (JMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the JMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the Junior Mathematical Olympiad and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	E	D	A	B	B	A	C	D	E	D	B	A	E	C	D	B	B	D	C	C	E	D	B	E

1. Which of the following calculations gives the largest answer?

A $1 - 2 + 3 + 4$ B $1 + 2 - 3 + 4$ C $1 + 2 + 3 - 4$ D $1 + 2 - 3 - 4$ E $1 - 2 - 3 + 4$

SOLUTION **A**

Since the numbers are the same in each sum, the largest answer results when the amount subtracted is the smallest. In option A the number 2 is subtracted, in option B it is 3, in option C it is 4, in option D both 3 and 4 are subtracted, and in option E both 2 and 3. So A gives the largest answer.

Alternatively, we can see directly that the sums have the following answers:

A: $1 - 2 + 3 + 4 = 6$
B: $1 + 2 - 3 + 4 = 4$
C: $1 + 2 + 3 - 4 = 2$
D: $1 + 2 - 3 - 4 = -4$
E: $1 - 2 - 3 + 4 = 0$

and therefore that A gives the largest answer.

2. It has just turned 22:22.

How many minutes are there until midnight?

A 178 B 138 C 128 D 108 E 98

SOLUTION **E**

There are $60 - 22 = 38$ minutes from 22:22 to 23:00 and then a further 60 minutes to midnight. Since $38 + 60 = 98$, there are 98 minutes to midnight.

FOR INVESTIGATION

2.1 How many seconds is it to midnight when my 24-hour clock, which shows hours, minutes and seconds, gives the time as 22:22:22?

2.2 What time does my clock show when it is 1000 minutes to midnight?

3. What is the value of $\dfrac{12\,345}{1+2+3+4+5}$?

A 1 B 8 C 678 D 823 E 12 359

SOLUTION **D**

In the context of the JMC we can answer the question by eliminating the options that cannot be correct without having to do a division sum to evaluate the fraction given in this question.

Since 12 345 and $1 + 2 + 3 + 4 + 5 = 15$ are both odd numbers, $\dfrac{12\,345}{1+2+3+4+5}$ must also be odd. So only options A, D and E could be correct. However, it is clear that

$$\frac{12\,345}{15} \neq 1 \text{ and } \frac{12\,345}{15} \neq 12\,359.$$

This rules out options A and E and leaves D as the correct option.

FOR INVESTIGATION

3.1 To verify that option D really is correct, we need to work out the answer to the division sum

$$12\,345 \div 15.$$

Because $15 = 3 \times 5$, the answer to this sum may be found by first dividing 12 345 by 3, and then dividing the answer by 5. Verify that the final answer is indeed 823.

3.2 The answer to Question 3 shows that

$$\frac{12\,345}{1+2+3+4+5}$$

is an integer. Which of the following are integers?

(a) $\dfrac{12}{1+2}$,

(b) $\dfrac{123}{1+2+3}$,

(c) $\dfrac{1234}{1+2+3+4}$,

(d) $\dfrac{123\,456}{1+2+3+4+5+6}$,

(e) $\dfrac{1\,234\,567}{1+2+3+4+5+6+7}$,

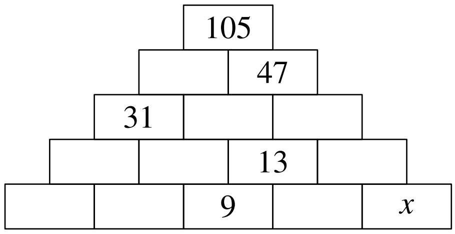
(f) $\dfrac{12\,345\,678}{1+2+3+4+5+6+7+8}$,

(g) $\dfrac{123\,456\,789}{1+2+3+4+5+6+7+8+9}$.

4. In this partly completed pyramid, each rectangle is to be filled with the sum of the two numbers in the rectangles immediately below it.

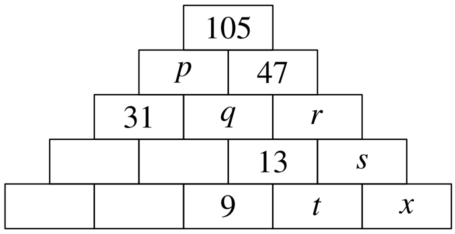
What number should replace x?

A 3 B 4 C 5 D 7 E 12



SOLUTION **A**

We have used p, q, r, s and t for the numbers in certain of the rectangles as shown in the diagram.



We now repeatedly use the fact that the number in each rectangle in the first four rows is the sum of the numbers in the rectangles immediately below it. This enables us to work out the values of p, q, r, s and t.

Applied to the top rectangle, this gives $105 = p + 47$. It follows that $p = 105 - 47 = 58$. Then, as $p = 31 + q$, we have $58 = 31 + q$. Therefore $q = 27$. Next, from $47 = q + r$,we deduce that $47 = 27 + r$. This gives $r = 20$. Next, from $r = 13 + s$, we have $20 = 13 + s$. Hence $s = 7$. We also have $13 = 9 + t$. Therefore $t = 4$. Finally, $s = t + x$. Therefore $7 = 4 + x$. We can now conclude that $x = 3$.

FOR INVESTIGATION

4.1 In the solution above to Question 4, we have found the value of x without finding the numbers in four of the rectangles. Complete the above diagram by finding these numbers.

5. The difference between $\frac{1}{3}$ of a certain number and $\frac{1}{4}$ of the same number is 3.

What is that number?

A 24 B 36 C 48 D 60 E 72

SOLUTION **B**

Since

$$\frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12},$$

it follows that $\frac{1}{12}$th of the number is 3. Therefore the number is $12 \times 3 = 36$.

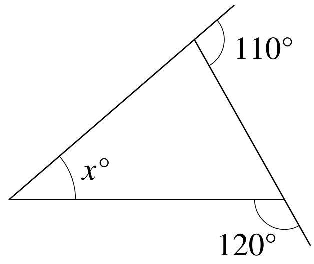
FOR INVESTIGATION

5.1 The difference between $\frac{1}{6}$ and $\frac{1}{8}$ of a certain number is 5. What is that number?

5.2 m and n are different non-zero numbers. The difference between $\frac{1}{m}$ and $\frac{1}{n}$ of the number p is k. Find a formula for p in terms of k, m and n.

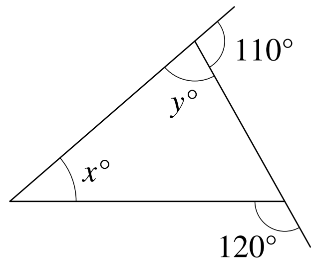
6. What is the value of x in this triangle?

A 45 B 50 C 55 D 60 E 65



SOLUTION **B**

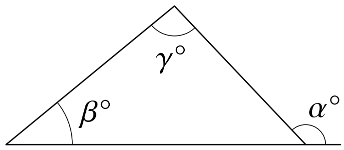
The angle marked on the diagram as $y°$ and the angle that is $110°$ are angles on a straight line. Therefore their sum is $180°$. It follows that $y° = 70°$. Therefore, by the Exterior Angle Theorem [see Problem 6.1 below], $120° = x° + y° = x° + 70°$. It follows that $x = 50$.



FOR INVESTIGATION

6.1 The *External Angle Theorem* says that the exterior angle of a triangle is the sum of the two opposite internal angles of the triangle.

In terms of the angles marked in the diagram, it says that



$$\alpha° = \beta° + \gamma°.$$

Give a proof of the Exterior Angle Theorem, using the fact that the sum of the angles on a straight line ($180°$) is the same as the sum of the angles in a triangle.

NOTE

The External Angle Theorem is also used in the solutions of Question 16 and Question 25.

7. The result of the calculation $123\,456\,789 \times 8$ is almost the same as $987\,654\,321$ except that two of the digits are in a different order.

What is the sum of these two digits?

A 3 B 7 C 9 D 15 E 17

SOLUTION **A**

METHOD 1

In the context of the JMC we are entitled to assume the truth of the statement that $123\,456\,789 \times 8$ is obtained by interchanging two of the digits of $987\,654\,321$. This leads to a quick way to answer the question without the need for a lot of arithmetic.

Because $9 \times 8 = 72$, the units digit of $123\,456\,789 \times 8$ is a 2. Starting from $987\,654\,321$, to obtain a 2 as the units digit we need to interchange the digits 1 and 2. So these are the two digits which are in a different order in the answer to the calculation. Now comes the easy bit, $1 + 2 = 3$.

METHOD 2

If we cannot take the statement in the question on trust, the only thing to do is to actually multiply $123\,456\,789$ by 8. If you do this you will see that the answer is $987\,654\,312$. It follows that it is the digits 1 and 2 that need to be interchanged.

Note that, in fact, as soon as we get as far as working out that $89 \times 8 = 712$ we can deduce that the digits 1 and 2 need to be interchanged. It is, however, necessary to do the whole sum to check that all the other digits are in the right order.

NOTE

The number $123\,456\,789 \times 8$ is a multiple of 8. We have the following test for whether a number is a multiple of 8.

An integer is a multiple of 8, if, and only if, its last three digits form a number which is a multiple of 8.

Since 321 is not a multiple of 8, this shows immediately that $987\,654\,321$ is not equal to $123\,456\,789 \times 8$.

FOR INVESTIGATION

7.1 Explain why an integer is a multiple of 2 if, and only if, its units digit is a multiple of 2.

7.2 Show that an integer is a multiple of 4 if, and only if, the number made up of its last 2 digits is a multiple of 4.

7.3 Show that an integer is a multiple of 8 if, and only if, the number made up of its last 3 digits is a multiple of 8.

7.4 Is $12\,345\,678$ a multiple of 8?

8. Which of the following has the same remainder when it is divided by 2 as when it is divided by 3?

A 3 B 5 C 7 D 9 E 11

SOLUTION **C**

The answer may be found by trying the options in turn. In this way we find that 7 has remainder 1 when divided by 2 and when divided by 3.

Alternatively, we see that all the numbers given as options are odd and so each has remainder 1 when divided by 2. So the correct option will be a number which also gives remainder 1 when divided by 3. It is easy to see that, of the given options, only 7 meets this requirement.

FOR INVESTIGATION

8.1 Find the smallest positive integer that has remainder 1 when divided by 3 and by 5.

8.2 Find the smallest positive integer that has remainder 3 when divided both by 5 and by 7.

8.3 Show that every positive integer which gives the same remainder when divided by 2 as when divided by 3 is either a multiple of 6 or one more than a multiple of 6.

8.4 Show that whenever p and q are different prime numbers with $p < q$, and k is an integer such that $0 \leq k < p$, then there is a positive integer which has remainder k both when divided by p and when it is divided by q.

8.5 Can you find a generalization of the fact you are asked to prove in 8.4?

9. According to a newspaper report, "A 63-year-old man has rowed around the world without leaving his living room." He clocked up 25 048 miles on a rowing machine that he received for his 50th birthday.

Roughly how many miles per year has he rowed since he was given the machine?

A 200 B 500 C 1000 D 2000 E 4000

SOLUTION **D**

The man is now 63 years old and was given the rowing machine for his 50th birthday. So he has had the rowing machine for 13 years and, possibly, a few months. Therefore the average number of miles per year that he has rowed is roughly

$$\frac{25\,048}{13} \approx \frac{26\,000}{13} = 2000.$$

Therefore, 2000 is roughly the number of miles per year that the man has rowed.

FOR INVESTIGATION

9.1 We are not told exactly how old the man is, but only that his exact age is between 63 years and 64 years. What is the corresponding range of values for the average number of miles he has rowed per year since his 50th birthday?

10. In the expression 1 ☐ 2 ☐ 3 ☐ 4 each ☐ is to be replaced by either + or ×.

What is the largest value of all the expressions that can be obtained in this way?

A 10 B 14 C 15 D 24 E 25

Solution **E**

In general, we obtain a larger number by multiplying two positive integers together than by adding them. The only exceptions are when one of the positive integers is 1, because $1+n > 1\times n$, and when both are 2, because $2 + 2 = 2 \times 2$.

Therefore to obtain the largest possible value we need to replace the first ☐ with + but the other two with ×. This leads us to the answer $1 + 2 \times 3 \times 4 = 1 + 24 = 25$.

For investigation

10.1 What is the largest value of all the expressions that can be obtained by replacing each ☐ in 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5 by either + or ×?

10.2 Show that if m and n are positive integers which are both greater than 1, then $m+n \leq m\times n$, and that, except when $m = n = 2$, $m + n < m \times n$.

11. What is the smallest prime number that is the sum of three different prime numbers?

A 11 B 15 C 17 D 19 E 23

Solution **D**

Consider three different prime numbers which include 2, say the prime numbers 2, p and q. Then p and q will both be odd numbers, and therefore $2 + p + q$ will be an even number greater than 2 and so cannot be a prime number. So, if we seek prime numbers that are sums of three different prime numbers, we need only consider sums of three different odd prime numbers.

The three smallest odd prime numbers are 3, 5 and 7, but their sum is 15 which is not prime. If we replace 7 by the next odd prime, 11, we have three odd primes with sum $3 + 5 + 11 = 19$, which is a prime number.

We cannot obtain a smaller prime number as a sum using 3 and two other odd prime numbers. If we do not include 3, the smallest sum of three odd prime numbers that we can obtain is $5 + 7 + 11 = 23$ which is greater than 19.

We can therefore deduce that 19 is the smallest prime number which is the sum of three different prime numbers.

For investigation

11.1 Find the smallest prime number which is greater than 23 and which is also the sum of three different prime numbers.

11.2 Find the smallest prime number which is the sum of *five* different prime numbers.

12. A fish weighs a total of 2 kg plus a third of its own weight.

What is the weight of the fish in kg?

A $2\frac{1}{3}$ B 3 C 4 D 6 E 8

SOLUTION **B**

METHOD 1

Since the fish weighs 2 kg plus one third of its weight, 2 kg is two thirds of its weight. Therefore one third of its weight is 1 kg, and so the total weight of the fish is 2 kg + 1 kg = 3 kg.

METHOD 2

We can also solve this problem using algebra.

We let x be the weight of the fish in kg. Now we use the information in the question to create an equation involving x that we can solve.

Because the fish weighs 2 kg plus one third of its weight,

$$x = 2 + \tfrac{1}{3}x.$$

It follows that

$$x - \tfrac{1}{3}x = 2,$$

and hence,

$$\tfrac{2}{3}x = 2.$$

Because $\frac{3}{2} \times \frac{2}{3} = 1$, we multiply both sides of this equation by $\frac{3}{2}$. In this way we deduce that

$$\begin{aligned} x &= \tfrac{3}{2} \times 2 \\ &= 3. \end{aligned}$$

FOR INVESTIGATION

12.1 A fish weighs 3 kg plus a fifth of its own weight. What is the weight of the fish in kg?

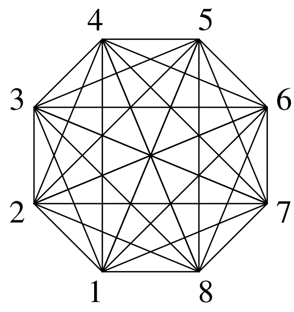
12.2 Find a formula, in terms of w and k, for the weight, in kg, of a fish that weighs w kg plus $\dfrac{1}{k}$ of its own weight.

12.3 Check that the formula that you found in answer to 12.2 gives the correct answers for Question 12 and Problem 12.1.

13. In the figure shown, each line joining two numbers is to be labelled with the sum of the two numbers that are at its end points.

How many of these labels are multiples of 3?

A 10 B 9 C 8 D 7 E 6



SOLUTION **A**

The figure consists of a regular octagon whose vertices are labelled with the positive integers from 1 to 8 inclusive. There is a line joining each pair of the vertices. It follows that the number of labels which are multiples of 3 is equal to the number of pairs of distinct integers in the range from 1 to 8, inclusive, whose sum is a multiple of 3.

The following table shows all multiples of 3 that can be a label, and for each multiple, the different ways of writing it as a sum of two distinct integers in the range from 1 to 8. We do not need to go beyond 15, as the largest integer that can appear as a label is $7 + 8 = 15$.

Multiple of 3	Sum(s)	Number of labels
3	$1 + 2$	1
6	$1 + 5$, $2 + 4$	2
9	$1 + 8$, $2 + 7$, $3 + 6$, $4 + 5$	4
12	$4 + 8$, $5 + 7$	2
15	$7 + 8$	1

It follows that the number of labels which are multiples of 3 is $1 + 2 + 4 + 2 + 1 = 10$.

FOR INVESTIGATION

13.1 How many pairs of distinct integers in the range from 1 to 8, inclusive, have a sum which is a multiple of 2?

13.2 How many pairs of distinct integers in the range from 1 to 9, inclusive, have a sum which is a multiple of 2?

13.3 How many pairs of distinct integers in the range from 1 to 10, inclusive, have a sum which is a multiple of 2?

13.4 Can you make a *conjecture* (that is, *an intelligent guess*) about a formula for the number of pairs of distinct integers in the range from 1 to n, inclusive, where n is a positive integer, which have a sum which is a multiple of 2? Can you *prove* that your formula is correct?

13.5 How many pairs of distinct integers in the range from 1 to 9, inclusive, have a sum which is a multiple of 3?

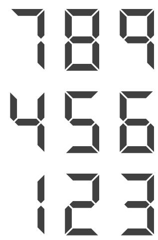
13.6 How many pairs of distinct integers in the range from 1 to 10, inclusive, have a sum which is a multiple of 3?

13.7 Can you conjecture, and then prove, a formula for the number of pairs of distinct integers in the range from 1 to n, inclusive, where n is a positive integer, which have a sum which is a multiple of 3?

14. Digits on a calculator are expressed by a number of horizontal and vertical illuminated bars. The digits and the bars which represent them are shown in the diagram.

How many digits are both prime and represented by a prime number of illuminated bars?

A 0 B 1 C 2 D 3 E 4



SOLUTION **E**

The digits that are primes are 2, 3, 5 and 7. [It is important to remember that 1 is *not* a prime number.] The numbers of illuminated bars used to represent them are

2: 5 bars; 3: 5 bars; 5: 5 bars; 7: 3 bars.

We see that each of them is represented by a prime number of bars. So there are 4 of the digits with the required property.

15. Which of the following is divisible by all of the integers from 1 to 10 inclusive?

A 23×34 B 34×45 C 45×56 D 56×67 E 67×78

SOLUTION **C**

It is easy to rule out four of the options using the fact that a product of integers is divisible by a prime number p if, and only if, at least one of the integers making up the product is divisible by p. [See Problem 15.2 for an example to show this statement is not in general true if p is not a prime number.]

Using this we see that, since neither 23 nor 34 is a multiple of 3, it follows that 23×34 is not a multiple of 3, since neither 34 nor 45 is a multiple of 7, 34×45 is not a multiple of 7, since neither 56 nor 67 is a multiple of 5, 56×67 is not a multiple of 5, and, similarly, 67×78 is not a multiple of 5.

This rules out the options A, B, D and E. In the context of the JMC this is enough for us to be able to conclude that option C is the correct answer.

However, for a full solution, we would need to check directly that 45×56 is divisible by all the integers from 1 to 10. This is straightforward.

Every integer is divisible by 1. Since 45 is divisible by 3, 5 and 9, so also is 45×56. Since 56 is divisible by 2, 4, 7 and 8, so also is 45×56. Since 45×56 is divisible both by 2 and by 3, which have no common factor, it is also divisible by 6. Since 45×56 is divisible by both 2 and by 5, which have no common factor, it is also divisible by 10. Therefore 45×56 is divisible by all the integers from 1 to 10, inclusive.

FOR INVESTIGATION

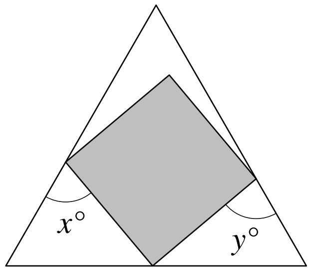
15.1 Which is the smallest positive integer that 45×56 is *not* divisible by?

15.2 Find a positive integer n such that neither 12 nor 30 is divisible by n, but their product, 12×30, is divisible by n. How many positive integers, n, with this property are there?

16. The diagram shows a square inside an equilateral triangle.

What is the value of $x + y$?

A 105 B 120 C 135 D 150 E 165



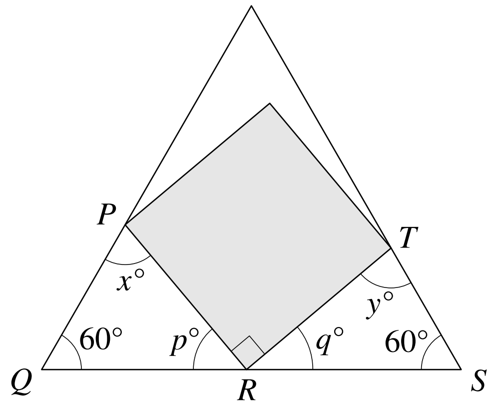
SOLUTION **D**

METHOD 1

We let P, Q, R, S and T be the points shown in the diagram. We also let $\angle QRP = p°$ and $\angle TRS = q°$.

Because it is an angle of a square, $\angle PRT = 90°$. Because they are angles of an equilateral triangle $\angle PQR = \angle RST = 60°$.

Because the angles of a triangle have sum 180°, from triangle PQR we have $x + p + 60 = 180$ and from triangle TRS, $y+q+60 = 180$. Therefore $x+p = 120$ and $y + q = 120$.



Because $\angle QRP$, $\angle PRT$ and $\angle TRS$ are angles on a straight line, $p + q + 90 = 180$ and therefore $p + q = 90$. It follows that $x + y = (x + p) + (y + q) - (p + q) = 120 + 120 - 90 = 150$.

METHOD 2

There is a quick method that it is all right to use in the context of the JMC, but which would *not* be acceptable if you had to give a full solution with detailed reasons.

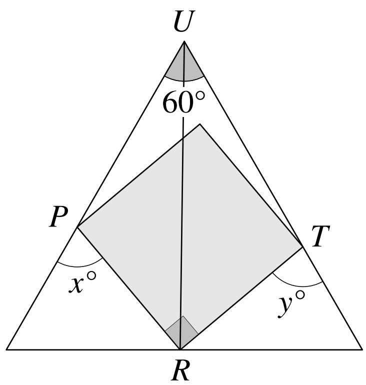
We have already shown that $p + q = 90$. Since the question does not give us individual values for p and q, we can assume that the answer is independent of their actual values. So, for simplicity, we assume that $p = q = 45$. Therefore in each of the triangles PQR and RST one of the angles is 60° and one is 45°. Therefore, because the sum of the angles in a triangle is 180°, both x and y are equal to $180 - 60 - 45 = 75$. We conclude that $x + y = 75 + 75 = 150$.

METHOD 3

Note that our answer, 150, as given above, is the sum, in degrees, of an angle of the square and an angle of the equilateral triangle.

The following methods shows, more directly, why this is so. We label the third vertex of the equilateral triangle as U and add the line joining R and U, as shown in the diagram.

We now apply the External Angle Theorem [see Problem 6.1, above] to the triangles PRU and RTU.



By applying the External Angle Theorem to the triangle PRU we obtain

$$x^\circ = \angle PUR + \angle PRU,$$

and, by applying this theorem to the triangle RTU,

$$y^\circ = \angle TUR + \angle TRU.$$

If we add these equations we obtain

$$\begin{aligned} x^\circ + y^\circ &= (\angle PUR + \angle PRU) + (\angle TUR + \angle TRU) \\ &= (\angle PUR + \angle TUR) + (\angle PRU + \angle TRU) \\ &= \angle PUT + \angle PRT \\ &= 60^\circ + 90^\circ \\ &= 150^\circ. \end{aligned}$$

17.

Knave of Hearts: “I stole the tarts.”
Knave of Clubs: “The Knave of Hearts is lying. ”
Knave of Diamonds: “The Knave of Clubs is lying.”
Knave of Spades “The Knave of Diamonds is lying.”

How many of the four Knaves were telling the truth?

A 1 B 2 C 3 D 4
E more information needed

SOLUTION **B**

Either the Knave of Hearts stole the tarts or he is innocent.

If the Knave of Hearts stole the tarts, he was telling the truth. So the Knave of Clubs was lying. Hence the Knave of Diamonds was telling the truth. Therefore the Knave of Spades was lying. So in this case two of the four Knaves were lying.

If the Knave of Hearts did not steal the tarts, he was lying. So the Knave of Clubs was telling the truth. Hence the Knave of Diamonds was lying. Therefore the Knave of Spades was telling the truth. So also in this case two of the four Knaves were lying.

We cannot tell from the information given whether or not the Knave of Hearts stole the tarts. But, as we have seen, we can be sure that, whether he stole them or not, two of the Knaves were telling the truth and two were lying.

18. Each of the fractions $\frac{2637}{18\,459}$ and $\frac{5274}{36\,918}$ uses the digits 1 to 9 exactly once.

The first fraction simplifies to $\frac{1}{7}$.

What is the simplified form of the second fraction?

A $\frac{1}{8}$ B $\frac{1}{7}$ C $\frac{5}{34}$ D $\frac{9}{61}$ E $\frac{2}{7}$

SOLUTION **B**

METHOD 1

We are told in the question that

$$\frac{2637}{18\,459} = \frac{1}{7}.$$

Now note that $5274 = 2 \times 2637$ and $36\,918 = 2 \times 18459$. It follows that, by cancelling the common factor 2 in the numerator and the denominator,

$$\frac{5274}{36\,918} = \frac{2 \times 2637}{2 \times 18\,459} = \frac{2637}{18\,459} = \frac{1}{7}.$$

METHOD 2

If you do not spot the quick method used above, there is nothing for it but to try out the options in turn.

We first consider option A. By cross multiplication we have

$$\frac{5274}{36\,918} = \frac{1}{8} \Leftrightarrow 5274 \times 8 = 36\,918 \times 1.$$

We can see that the equation $5274 \times 8 = 36\,918 \times 1$ cannot be correct just by looking at the units digits on the two sides of the equation. On the left hand side 4×8 gives a units digit of 2, but on the right hand side 8×1 gives a units digit of 8. So the equation is not correct. We deduce that

$$\frac{5274}{36\,918} \neq \frac{1}{8}$$

and so option A is not the correct one.

Next we look at option B. Using cross multiplication again, we have

$$\frac{5274}{36\,918} = \frac{1}{7} \Leftrightarrow 5274 \times 7 = 36\,918 \times 1.$$

It is straightforward to check that the equation $5274 \times 7 = 36\,918$ is true. Therefore option B is correct.

FOR INVESTIGATION

18.1 Check that $5274 \times 7 = 36\,918 \times 1$.

18.2 The options C, D and E can be ruled out using an argument involving the units digits, just as we did above for option A. Show how this can be done.

19. One of the following cubes is the smallest cube that can be written as the sum of three positive cubes.

Which is it?

A 27 B 64 C 125 D 216 E 512

SOLUTION **D**

The positive cubes are the numbers in the sequence 1, 8, 27, 64, 125, 216, 343, ...

It is straightforward to check that none of the first fives cubes in this sequence is the sum of three smaller positive cubes.

For example, as $27 + 27 + 27 = 81$, and $81 < 125$, any three cubes with sum 125 must include 64 at least once. The three cubes couldn't include 64 twice because $64 + 64 > 125$. However, if we had $p + q + 64 = 125$, where p and q are positive cubes which are smaller then 64, then $p + q = 125 - 64 = 61$, which is impossible as the only values p and q can take are 1, 8 and 27. So 125 is not the sum of three positive cubes.

However, $216 = 27 + 64 + 125$, and so 216 is the sum of three positive cubes and so is smallest cube that can be written as the sum of three positive cubes.

FOR INVESTIGATION

19.1 Show that none of 1, 8, 27 and 64 is the sum of three smaller positive cubes.

19.2 Find the next smallest cube that can be written as the sum of three positive cubes.

19.3 Find the smallest cube that can be written as the sum of three positive cubes *in more than one way*.

19.4 Problem 19.3 brings to mind a famous story about the great Indian mathematician Srinivasa Ramanujan. Ramanujan came to England in 1914 to work with the English mathematician G. H. Hardy. Hardy, in his obituary of Ramanujan, relates that he went to see Ramanujan, when he was lying ill in Putney. He goes on "I had ridden in taxi-cab No. 1729, and remarked that the number ($7 \times 13 \times 19$) seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No', he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' I asked him, naturally, whether he knew the answer to the corresponding problem for fourth powers; and he replied after a moment's thought, that he could see no obvious example, and he thought that the first such number must be very large."

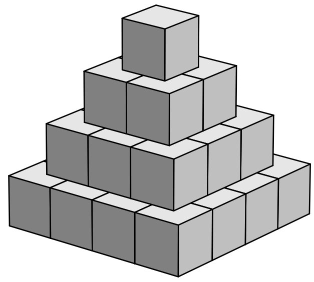
Find two different ways of expressing the number 1729 as the sum of two cubes.

19.5 Find the smallest positive integer that can be expressed as the sum of two fourth powers in two different ways. [Ramanujan was right about the number being large. Unless you can find the answer in a book or on the web, you will probably need a computer program to find it. Note, however, that the great Swiss mathematician Leonhard Euler (1707-1783) found the answer 200 years before electronic calculators became available.]

19.6 The number, 1729, mentioned in Problem 19.4, is not itself a cube. Indeed, it follows from *Fermat's Last Theorem* that there are no positive cubes that can be expressed as the sum of two positive cubes. Find (in a book or on the web) a statement of Fermat's Last Theorem, and the name of the first mathematician to give a proof of it.

20. The diagram shows a pyramid made up of 30 cubes, each measuring $1\text{ m} \times 1\text{ m} \times 1\text{ m}$.

What is the total surface area of the whole pyramid (including its base)?

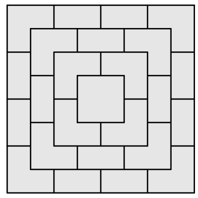


A 30 m^2 B 62 m^2 C 72 m^2
D 152 m^2 E 180 m^2

SOLUTION **C**

The surface area that can be seen by looking up from below the pyramid is that of $4 \times 4 = 16$ squares each measuring $1\text{ m} \times 1\text{ m}$. So the surface area of the base is 16 m^2,

The view looking down from above the pyramid is shown in the diagram. The surface area that can be seen is made up of a complete $1\text{ m} \times 1\text{ m}$ square, some three-quarter squares and some half squares. Without adding these up, we can see that the surface area is also 16 m^2.



The view from each of the four sides is made up of a pyramid of 10 squares whose surface area is 10 m^2.

Therefore, the total surface area is $2 \times 16\text{ m}^2 + 4 \times 10\text{ m}^2 = 72\text{ m}^2$.

FOR INVESTIGATION

20.1 What is the total surface area of a similar pyramid with 5 layers, with the bottom layer consisting of a 5×5 array of cubes, each measuring $1\text{ m} \times 1\text{ m} \times 1\text{ m}$?

20.2 Find a formula in terms of n for the surface area of a similar pyramid made up of n layers, with the bottom layer consisting of an $n \times n$ array of cubes, each measuring $1\text{ m} \times 1\text{ m} \times 1\text{ m}$.

20.3 Check that the formula that you obtained as your answer to 20.2 gives the correct values for $n = 1, 2, 3, 4, 5$.

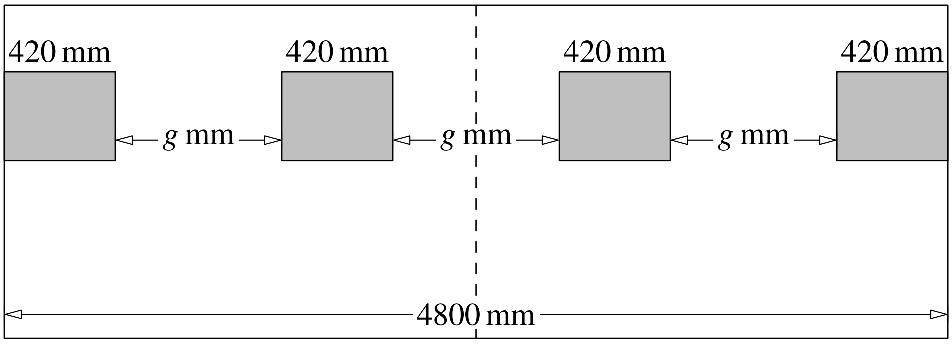
21. Gill is now 27 and has moved into a new flat. She has four pictures to hang in a horizontal row on a wall which is 4800 mm wide. The pictures are identical in size and are 420 mm wide. Gill hangs the first two pictures so that one is on the extreme left of the wall and one is on the extreme right of the wall. She wants to hang the remaining two pictures so that all four pictures are equally spaced.

How far should Gill place the centre of each of the two remaining pictures from a vertical line down the centre of the wall?

A 210 mm B 520 mm C 730 mm D 840 mm E 1040 mm

SOLUTION **C**

Let the three gaps between the pictures each be g mm wide.



Since each of the four pictures is 420 mm wide and the wall is 4800 mm wide,

$$4 \times 420 + 3g = 4800,$$

and therefore

$$\begin{aligned} 3g &= 4800 - 4 \times 420 \\ &= 4800 - 1680 \\ &= 3120. \end{aligned}$$

It follows that

$$\begin{aligned} g &= \tfrac{1}{3} \times 3120 \\ &= 1040. \end{aligned}$$

The distance between the centres of the middle two pictures is equal to the width of one picture and the width of the gap, that is, in mm, $420 + g = 420 + 1040 = 1460$. The distance between the centre of one of these pictures and the centre line is half this distance. Therefore the required distance is, in mm,

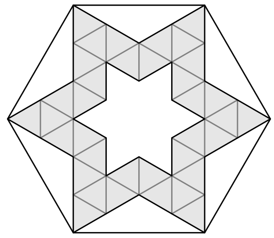
$$\tfrac{1}{2} \times 1460 = 730.$$

22. The diagram shows a shaded region inside a regular hexagon.

The shaded region is divided into equilateral triangles.

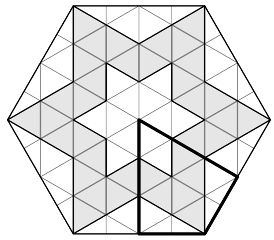
What fraction of the area of the hexagon is shaded?

A $\frac{3}{8}$ B $\frac{2}{5}$ C $\frac{3}{7}$ D $\frac{5}{12}$ E $\frac{1}{2}$



Solution **E**

We form a complete grid inside the hexagon, as shown in the figure.



In this way the hexagon is divided up into a number of congruent equilateral triangles and, around the edge, some triangles each congruent to half of the equilateral triangles.

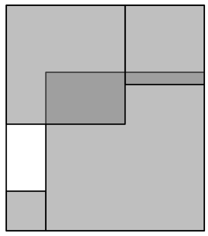
We could now use the grid to work out the shaded and unshaded areas in terms of the areas of the equilateral triangles, and hence work out which fraction of the area of the hexagon is shaded.

It is a little easier to exploit the sixfold symmetry of the figure and just work out the fraction of the area surrounded by the heavy lines that is shaded.

We see that in this part of the hexagon there are six shaded equilateral triangles, four unshaded equilateral triangles and four unshaded triangles whose areas are each half that of the equilateral triangles. So the unshaded area is equal to that of six of the equilateral triangles. It follows that the shaded area is equal to the unshaded area.

We conclude that the fraction of the hexagon that is shaded is $\frac{1}{2}$.

23. The diagram shows four shaded glass squares, with areas of $1\,\text{cm}^2$, $4\,\text{cm}^2$, $9\,\text{cm}^2$ and $16\,\text{cm}^2$, placed in the corners of a rectangle. The largest square overlaps two others. The area of the region inside the rectangle but not covered by any square (shown unshaded) is $1.5\,\text{cm}^2$.



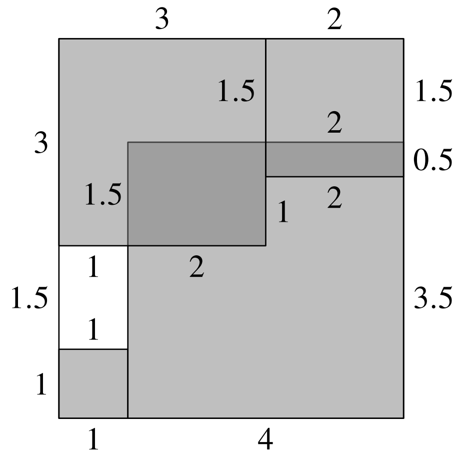
What is the area of the region where the squares overlap (shown dark grey)?

A $2.5\,\text{cm}^2$ B $3\,\text{cm}^2$ C $3.5\,\text{cm}^2$ D $4\,\text{cm}^2$ E $4.5\,\text{cm}^2$

Solution **D**

Method 1

The squares in the corners of the large rectangle are of sizes $1\,\text{cm} \times 1\,\text{cm}$, $2\,\text{cm} \times 2\,\text{cm}$, $3\,\text{cm} \times 3\,\text{cm}$ and $4\,\text{cm} \times 4\,\text{cm}$.



The white rectangle has width 1 cm and area $1.5\,\text{cm}^2$. It follows that it has height 1.5 cm.

We can now deduce that the large rectangle has height 5.5 cm and width 5 cm, and hence that the lengths are, in cm, as shown in the diagram.

We therefore see that the region shown dark grey is made up of two rectangles, one with width 2 cm and height 1.5 cm, and the other with width 2 cm and height 0.5 cm.

Therefore the area of this region is $(2 \times 1.5)\,\text{cm}^2 + (2 \times 0.5)\,\text{cm}^2 = 3\,\text{cm}^2 + 1\,\text{cm}^2 = 4\,\text{cm}^2$.

Method 2

Once we have shown, as above, that the large rectangle has height 5.5 cm and width 5 cm, we can find the area of the overlap without finding the dimensions of the two rectangles that make it up. Instead we can give an argument just in terms of areas, as follows.

The area of the large rectangle is $5.5\,\text{cm} \times 5\,\text{cm} = 27.5\,\text{cm}^2$. Since the area not covered by any of the squares is $1.5\,\text{cm}^2$, the area of the large rectangle covered by the squares is $27.5\,\text{cm}^2 - 1.5\,\text{cm}^2 = 26\,\text{cm}^2$.

The total area of the squares is $1\,\text{cm}^2 + 4\,\text{cm}^2 + 9\,\text{cm}^2 + 16\,\text{cm}^2 = 30\,\text{cm}^2$.

The difference between these two areas is accounted for by the overlap. Therefore the area of the overlap is $30\,\text{cm}^2 - 26\,\text{cm}^2 = 4\,\text{cm}^2$.

24. A *palindromic number* is a number which reads the same when the order of its digits is reversed.

What is the difference between the largest and smallest five-digit palindromic numbers that are both multiples of 45?

A 9180 B 9090 C 9000 D 8910 E 8190

SOLUTION **B**

We use the notation '$abcde$' for the number which is represented by the digits a, b, c, d and e when expressed using the standard base 10. Using this notation we can write a five-digit palindromic number as '$abcba$', where a, b and c are digits.

Since $45 = 5 \times 9$, and 5 and 9 have no common factors, the five-digit palindromic number '$abcba$' is a multiple of 45 if, and only if, it is a multiple of both 5 and 9.

A number is a multiple of 5 if, and only if, its units digit is 0 or 5. Here the units digit a cannot be 0, since otherwise '$abcba$' would not be a five-digit number. We deduce that a is 5. Thus a five-digit palindromic number which is divisible by 5 is of the form '$5bcb5$', where b and c are digits.

A number is a multiple of 9 if, and only if, the sum of its digits is a multiple of 9.

The smallest five-digit palindromic number that is a multiple of 45 has the form '$50c05$', corresponding to taking b to be 0. The digit sum of '$50c05$' is $10 + c$. For this to be a multiple of 9 we need to take the digit c to be 8. Therefore 50805, with digit sum 18, is the smallest five-digit palindromic number which is a multiple of 45.

The largest five-digit palindromic number that is a multiple of 45 has the form '$59c95$', corresponding to taking b to be 9. The digit sum of '$59c95$' is $28 + c$. For this to be a multiple of 9 we need to take the digit c to be 8. Therefore 59895, with digit sum 36, is the largest five-digit palindromic number which is a multiple of 45.

The difference between these two numbers is $59895 - 50805 = 9090$.

FOR INVESTIGATION

24.1 Find the largest and smallest six-digit palindromic numbers that are multiples of 45. What is their difference?

24.2 In the solution of Question 24 we have used the following fact:

> (1). A positive integer is a multiple of by 9 if, and only if, the sum of its digits is a multiple of 9.

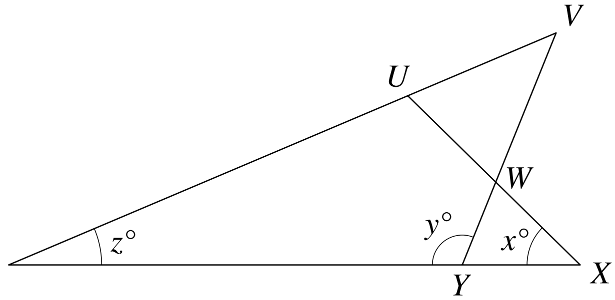
This is a consequence of the more general fact:

> (2). The remainder when a positive integer is divided by 9 is equal to the remainder when the sum of its digits is divided by 9.

Explain why (2) is true, and why (1) follows from (2).

24.3 What is the test, in terms of the sum of its digits, for whether a positive integer is a multiple of 3?

25. The four straight lines in the diagram are such that $VU = VW$. The sizes of $\angle UXZ$, $\angle VYZ$ and $\angle VZX$ are $x°$, $y°$ and $z°$.



Which of the following equations gives x in terms of y and z?

A $x = y - z$ B $x = 180 - y - z$ C $x = y - \frac{z}{2}$

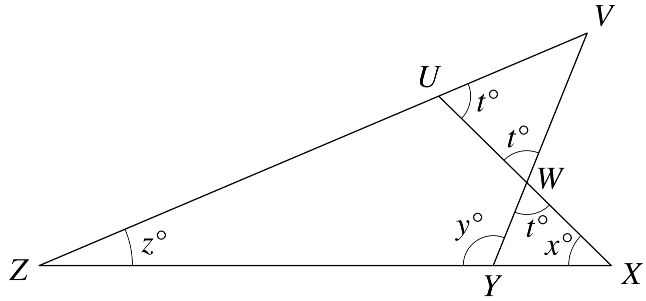
D $x = y + z - 90$ E $x = \frac{y - z}{2}$

Solution **E**

Because $VU = VW$, the triangle VUW is isosceles and so $\angle VUW$ and $\angle VWU$ are equal.

Because they are vertically opposite $\angle VWU$ and $\angle YWX$ are equal.

Therefore $\angle VUW$, $\angle VWU$ and $\angle YWX$ are all equal. We let the size of these three angles be $t°$, as marked on the diagram.



We now apply the *Exterior Angle Theorem* [see Problem 6.1, above] in turn to the exterior angle, $\angle VUW$ of triangle UZX and the exterior angle $\angle ZYW$ of triangle WYX. This gives

$$t = z + x$$

and

$$y = t + x.$$

Therefore, using the first equation to substitute $z + x$ for t in the second equation, we have

$$\begin{aligned} y &= (z + x) + x \\ &= z + 2x. \end{aligned}$$

Hence, by rearranging the last equation,

$$2x = y - z.$$

By dividing both sides of the last equation by 2, we conclude that

$$x = \frac{y - z}{2}.$$