UK JUNIOR MATHEMATICAL CHALLENGE
May 6th 2011

SOLUTIONS

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some ***Extension Problems*** for further investigations.

The Junior Mathematical Challenge (JMC) is a multiple choice contest, in which you are presented with five alternative answers, of which just one is correct. No reasons for the answers need to be given. However, providing good and clear explanations is the heart of doing Mathematics. So here we have aimed to give full solutions with all steps explained. We therefore hope that these solutions can be used as a model for the type of written solution that is expected in the Junior Mathematical Olympiad and similar competitions.

We welcome comments on these solutions, and, especially, corrections or suggestions for improving them. Please send your comments,

either by e-mail to enquiry@ukmt.co.uk, or by post to JMC Solutions, UKMT Maths Challenges Office, School of Mathematics, University of Leeds, Leeds LS2 9JT.

Quick Marking Guide

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	E	B	C	B	E	D	A	A	C	C	E	E	A	A	B	D	C	D	A	B	D	E	D	C

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1. What is the value of $2 \times 0 \times 1 + 1$?

A 0   B 1   C 2   D 3   E 4

Solution: **B**

This is simply a matter of doing the sum: $2 \times 0 \times 1 + 1 = 0 \times 1 + 1 = 0 + 1 = 1$.

2. How many of the integers 123, 234, 345, 456, 567 are multiples of 3?

A 1   B 2   C 3   D 4   E 5

Solution: **E**

We could simply divide each of the given numbers by 3, and check that there is no remainder in each case. It is quicker to use the fact that

An integer is divisible by 3 if and only if the sum of its digits is a multiple of 3.

Now $1+2+3=6$, $2+3+4=9$, $3+4+5=12$, $4+5+6=15$ and $5+6+7=18$, which are all multiples of 3. So 123, 234, 345, 456 and 567 are all multiples of 3.

Extension Problems

The fact that an integer is divisible by 3 if and only if the sum of its digits is a multiple of 3, follows from the fact that

The remainder when a positive integer is divided by 9 *is equal to the remainder when the sum of its digits is divisible by* 9.

For example, consider the number 2011. The sum of its digits is 4, which leaves remainder 4 when we divide by 9. We can deduce that 2011 has remainder 4 when we divide by 9. When the sum of the digits of a number is itself quite large we can repeat the process.

For example, consider the number $777...777$ with 2011 7s. What is the remainder when this number is divided by 9? The sum of its digits is $2011 \times 7 = 14077$. The sum of the digits of 14077 is 19, which has remainder 1 when divided by 9. We can deduce that the number $777...777$ has remainder 1 when divided by 9.

2.1 What is the remainder when the following numbers are divided by 9?

a) 123456789;   b) 9876543210;   c) 555…555 with 2011 5s.

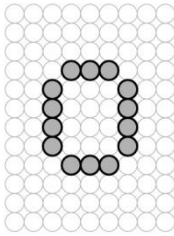
2.2 Can you prove the above fact about the remainder when a positive integer is divided by 9?

2.3 We didn’t really need to work out all the sums of the digits of the given numbers. The digits of each of these numbers are three successive integers. So all we need do is to make use of the fact that

The sum of three successive integers is always a multiple of 3.

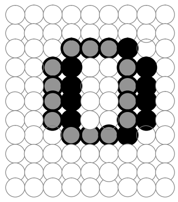
Can you prove this?

3. A train display shows letters by lighting cells in a grid, such as the letter 'o' shown. A letter is made **bold** by also lighting any unlit cell immediately to the right of one in the normal letter. How many cells are lit in a **bold** 'o'?



A 22 B 24 C 26 D 28 E 30

Solution: **B**



The only thing to do here is to draw (or imagine) the figure corresponding to a **bold** 'o', and to count the number of cells that are lit. We see that the extra cells that need to be lit are then 10 cells shown in black. This makes 24 lit cells altogether.

4. The world's largest coin, made by the Royal Mint of Canada, was auctioned in June 2010. The coin has mass 100 kg, whereas a standard British £1 coin has mass 10 g. What sum of money in £1 coins would weigh the same as the record-breaking coin?

A £100 B £1000 C £10 000 D £100 000 E £1 000 000

Solution: **C**

We have to work out how many £1 coins, each weighing 10 g, we need to get a total weight of 100 kg. Now 1 kg = 1000 g, and so 100 kg = 100,000 g. So we need $\frac{100{,}000}{10} = 10{,}000$ of these coins. That is, £10 000 in money.

5. All old Mother Hubbard had in her cupboard was a Giant Bear chocolate bar. She gave each of her children one-twelfth of the chocolate bar. One third of the bar was left. How many children did she have?

A 6 B 8 C 12 D 15 E 18

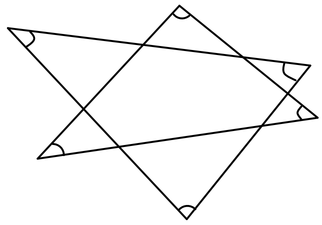
Solution: **B**

One third of the bar was left, so Mother Hubbard's children ate two-thirds of the bar. Since they ate one-twelfth of the bar each, Mother Hubbard had

$$\frac{2/3}{1/12} = \frac{2}{3} \times \frac{12}{1} = \frac{24}{3} = 8 \text{ children.}$$

6. What is the sum of the marked angles in the diagram?

A 90° B 180° C 240° D 300° E 360°



Solution: **E**

The six marked angles are the interior angles of 2 triangles. The interior angles of 1 triangle add up to 180°. So the marked angles add up to $2 \times 180^\circ = 360^\circ$.

7. Peter Piper picked a peck of pickled peppers. 1 peck = $\frac{1}{4}$ bushel and 1 bushel = $\frac{1}{9}$ barrel. How many **more** pecks must Peter Piper pick to fill a barrel?

A 12 B 13 C 34 D 35 E 36

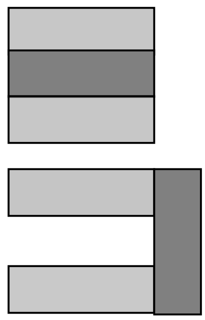
Solution: **D**

We have 1 barrel = 9 bushels and 1 bushel = 4 pecks, and so 1 barrel = $9 \times 4 = 36$ pecks. Peter Piper has already picked 1 peck, so he needs to pick 35 more pecks to fill a barrel.

8. A square is divided into three congruent rectangles. The middle rectangle is removed and replaced on the side of the original square to form an octagon as shown.

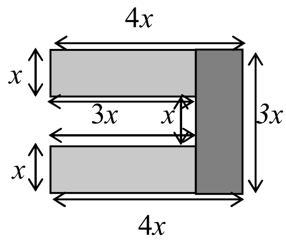
What is the ratio of the length of the perimeter of the square to the length of the perimeter of the octagon?

A 3:5 B 2:3 C 5:8 D 1:2 E 1:1



Solution: **A**

Suppose that the square has side length $3x$. Then the square has perimeter of length $4 \times 3x = 12x$.



Each rectangle has size $3x \times x$. So we see that the octagon has a perimeter length

$$4x + 3x + 4x + x + 3x + x + 3x + x = 20x.$$

Hence ratio of the lengths is $12x : 20x = 3 : 5$

9. What is the smallest possible difference between two different 9-digit integers, each of which includes all the digits 1 to 9?

A 9 B 18 C 27 D 36 E 45

Solution: **A**

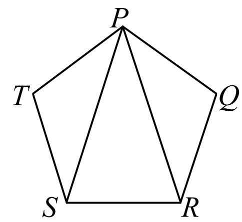
Let x and y be two different numbers each made up of the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9, with $x > y$. To make $x - y$ as small as possible, it would be best if they differed just in the units digit, but, clearly, this is not possible. So we aim to find x and y which are the same except for their tens and units digits. These could be 21 and 12, for example with $x = 987654321$ and $y = 987654312$. This gives $x - y = 9$.

Can we do any better? We now will the fact that we gave in the solution to Question 2: *The remainder when a positive integer is divided by* 9 *is equal to the remainder when the sum of its digits is divisible by* 9 to show we cannot get a smaller difference. For, as x and y are made up of the same digits which add up to 45, they have the remainder 0 when we divide them by 9. So $x - y$ also has remainder 0 when we divide by 9. So, as $x \neq y$, 9 is the least possible value for $x - y$.

Extension Problem

9.1 What is the *largest* difference between two 9-digit integers whose digits use each of the digits 1 to 9 once and once only?

10. You want to draw the shape on the right without taking your pen off the paper and without going over any line more than once? Where should you start?



A only at T or Q B only at P C only at S and R

D at any point E the task is impossible

Solution: **C**

There are several ways to draw the diagram starting at R without taking the pen off the paper or going over a line more than once. Here is one example of such a path:

$$R \to P \to S \to R \to Q \to P \to T \to S.$$

If we reverse this path, we have a way to draw the diagram starting at S.

How can we be sure that there are no other possible starting points? Notice that each time the path goes through a vertex it uses up two of the edges that have that vertex as one of their endpoints. We use the technical term *degree* of a vertex, for the number of edges that have that vertex as one of its endpoints. Since the path uses each edge just once, the degree of each vertex must be even (twice the number of times the path goes through the vertex), except for the case where the path starts at one vertex and ends at a different vertex, when these two vertices have degrees that are odd numbers (twice the number of times the path goes through the vertex plus one for the initial or final edge of the path). The degrees of the five vertices in the diagram are shown in the table.

vertex	*degree*
P	4
Q	2
R	3
S	3
T	2

We see that there are just two vertices, *R* and *S*, whose degrees are odd numbers. So *R* and *S* must be the endpoints of any path which goes along each edge once, and drawing which does not involve taking your pen off the paper.

We have seen that a *necessary* condition that we can draw a diagram made up of lines joining points in this way is that *there are at most two vertices whose degrees are odd numbers*.

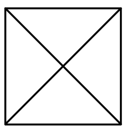
In fact, it turns out that this condition is also sufficient. So we can draw a diagram of this kind without taking our pen off the paper and without going over any line more than once, if and only if, there are at most two vertices whose degrees are odd numbers. The proof of this, while not very difficult, takes some care.

This problem has its origin in the Königsberg Bridge Problem. You will find it discussed in many books; for example, in Robin Wilson's book *Introduction to Graph Theory* where it comes under the heading of Eulerian graphs.

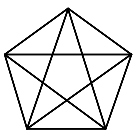
Extension Problems

10.1 Which of the following diagrams can be drawn without taking your pen off the paper and without going over any line more than once? In which cases is it possible to finish at the same point as where you began the drawing?

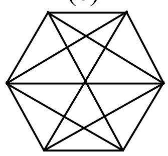
(a)



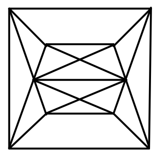
(b)



(c)

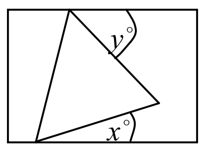


(d)



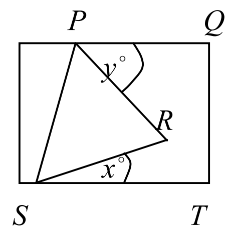
11. The diagram shows an equilateral triangle inside a rectangle. What is the value of $x + y$?

A 30 B 45 C 60 D 75 E 90



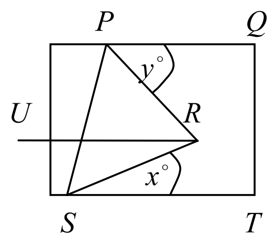
Solution: **C**

The marked angles are two angles of the pentagon *PRSTQ*. The angles at Q and T are both 90°. The interior angles of the equilateral triangle are all 60°. Hence the angle in the pentagon at R is 300°. The sum of the angles in a pentagon is 540°. So $x + y + 300 + 90 + 90 = 540$. Therefore $x + y = 60$.



An alternative method is to add a line *RU* through *R* and parallel to *ST*, as shown.

Because *RU* is parallel to *PQ* and *ST*, we have that $x^\circ + y^\circ = \angle SRU + \angle URP = \angle SRP = 60^\circ$.



12. If ▲ + ▲= ■ and ■ + ▲ = ● and ♦= ● + ■ + ▲, how many ▲s are equal to ♦?

A 2 B 3 C 4 D 5 E 6

Solution: **E**

We have that ■ = ▲ + ▲ (1), and ● = ■ + ▲ (2). Substituting from (1) into (2) gives ● =▲ + ▲ + ▲. Hence from ♦= ● + ■ + ▲, we deduce that ♦= (▲ + ▲ + ▲) + (▲ + ▲) + ▲. So ♦ is equal to six ▲s.

13. What is the mean of $\frac{2}{3}$ and $\frac{4}{9}$?

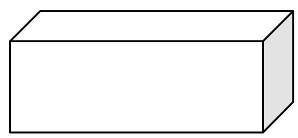
A $\frac{1}{2}$ B $\frac{2}{9}$ C $\frac{7}{9}$ D $\frac{3}{4}$ E $\frac{5}{9}$

Solution: **E**

The mean of two numbers, a and b, is the average $\frac{1}{2}(a+b)$. Hence the mean of $\frac{2}{3}$ and $\frac{4}{9}$ is

$$\frac{1}{2}\left(\frac{2}{3}+\frac{4}{9}\right)=\frac{1}{2}\left(\frac{6+4}{9}\right)=\frac{1}{2}\left(\frac{10}{9}\right)=\frac{5}{9}.$$

14. The diagram shows a cuboid in which the area of the shaded face is one-quarter of the area of each of the two visible unshaded faces. The total surface area of the cuboid is 72 cm². What, in cm², is the area of one of the visible unshaded faces of the cuboid?



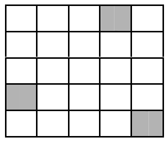
A 16 B 28.8 C 32 D 36 E 48

Solution: **A**

Let the area of the shaded face be x cm^2. Then each of the visible unshaded faces has area $4x$ cm^2. Hence the total surface area of the cuboid in cm^2 is $2(4x+4x+x)=18x$. So $18x=72$. Therefore $x=4$, and the area of one of the visible shaded faces, in cm^2 is $4x=16$.

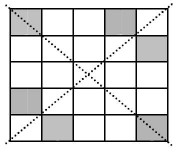
15. What is the smallest number of *additional* squares which must be shaded so that this figure has at least one line of symmetry *and* rotational symmetry of order 2 ?

A 3 B 5 C 7 D 9 E more than 9



Solution: **A**

In the diagram on the right we have shaded the three additional squares that must be shaded if the square is to have rotational symmetry of order 2. With these squares shaded the square has both the diagonals as lines of symmetry.



So the smallest number of additional squares that need to be shaded is 3.

16. The pupils in Year 8 are holding a mock election. A candidate receiving more votes than any other wins. The four candidates receive 83 votes between them. What is the smallest number of votes the winner could receive?

A 21 B 22 C 23 D 41 E 42

Solution: **B**

Since $\frac{1}{4}(83) = 20.75$, the winner must get at least 21 votes. If one candidate gets 21 votes there are 62 votes to be shared between the other three candidates, and so, as $3 \times 20 < 62$ one of these candidates must get at least 21 votes. So a candidate with 21 votes cannot be the winner.

However if one candidate gets 22 votes and the others 21, 20 and 20 votes respectively, all 83 votes have been used, and the winner receives only 22 votes.

17. Last year's match at Wimbledon between John Isner and Nicolas Malut, which lasted 11 hours and 5 minutes, set a record for the longest match in tennis history. The fifth set of the match lasted 8 hours and 11 minutes.

Approximately what fraction of the whole match was taken up by the fifth set?

A $\frac{1}{5}$ B $\frac{2}{5}$ C $\frac{3}{5}$ D $\frac{3}{4}$ E $\frac{9}{10}$

Solution: **D**

We have that 8 hours and 11 minutes is $8 \times 60 + 11 = 491$ minutes and 11 hours and 5 minutes is $11 \times 60 + 5 = 665$ minutes. Therefore the exact fraction taken by the last set is $\frac{491}{665}$. So we need to decide which of the given fractions this is closest to. This can be done in more than one way. For example, as 491 is just below 500 and 665 is almost $\frac{1}{3}$rds of 2000, we have that

$\frac{491}{665} \sim \frac{500}{\frac{1}{3}(2000)} = \frac{1}{\frac{1}{3}(4)} = \frac{3}{4}$. (Here "~" means "is approximately equal to".)

Another method is to say that $\frac{491}{665} \sim \frac{490}{665} = \frac{98}{135} = \frac{14}{19} \sim \frac{15}{20} = \frac{3}{4}$.

18. Peri the winkle leaves on Monday to go and visit Granny, 90m away. Except for rest days, Peri travels 1m each day (24-hour period) at a constant rate and without pause. However, Peri stops for a 24-hour rest every tenth day, that is, after nine days travelling. On which day does Peri arrive at Granny's? A Sunday B Monday C Tuesday D Wednesday E Thursday

Solution: **C**

Until Peri reahes Granny's, he travels 9m in 9 days and then rests for a day. So he travels 9m in 10 days. So it takes him 90 days to travel the first $9 \times 9 = 81$m. Then after a further 9 days he has travelled the final 9m and has reached Granny's. So the journey takes him 99 days. Now $99 = 14 \times 7 + 1$, so the journey takes him 14 weeks and 1 day. Therefore he arrives on a Tuesday.

19. A list is made of every digit that is the units digit of at least one prime number. How many of the following numbers appear in the list? A 1 B 2 C 3 D 4 E 5

Solution: **D**

The numbers 2, 3 and 5 are themselves prime numbers so they occur in this list. Also 1 is the units digit of the prime number 11 (and also 31, 41 and many more). However a number with units digit 4 is an even number greater than 2 and so is not prime. Thus, there are 4 numbers in the list, namely 1, 2, 3 and 5 which are the units digits of at least one prime number.

The German mathematician Peter Dirichlet (1805-1855) proved a theorem from which it follows that there are infinitely many prime numbers in the sequence 1, 11, 21, 31, 41, … of numbers which have 1 as their units digit.

Extension Problems

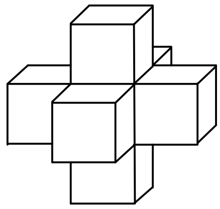
19.1 Which of the other digits, 0, 6, 7, 8 and 9 are the units digits of at last one prime number?

19.2 Use the web to find out about Peter Dirichlet and his theorem about prime numbers in arithmetic sequences. The proof of the theorem involves some very advancved mathematics, but the statement of the theorem is not hard to understand. The website:

http://www-history.mcs.st-and.ac.uk/

is an excellent source for the biographies of mathematicians.

20. One cube has each of its faces covered by one face of an identical cube, making a solid as shown. The volume of the solid is 875 cm^3. What, in cm^2, is the surface area of the solid?



A 750 B 800 C 875 D 900 E 1050

Solution: **A**

The solid is made up of 7 cubes. So each of them has volume $\frac{1}{7}(875) = 125 = 5^3$ cm^3. Hence the side length of each of the cubes is 5 cm. Each of the 6 faces of the central cube is covered by one face of one of the other 6 cubes, each of which has 5 faces showing. So the surface of the figure is made up of $6 \times 5 = 30$ squares, each with side length 5 cm and hence area 25 cm^2. Hence the total area of the figure is $30 \times 25 = 750\,cm^2$.

21. Gill leaves Lille by train at 09:00. The train travels the first 27 km at 96 km/h. It then stops at Lens for 3 minutes before travelling the final 29 km to Lillers at 96 km/h. At what time does Gill arrive at Lillers?

A 09:35 B 09:38 C 09:40 D 09:41 E 09:43

Solution: **B**

Gill travels $27 + 29 = 56$ km at 96 km/h. This takes $\frac{56}{96} = \frac{7}{12}$ th of an hour, that is, $\frac{7}{12} \times 60 = 35$ minutes. The train stops at Lens for 3minutes, so the journey takes 38 minutes all together. So Gill arrives at Lillers at 09:38.

22. Last week Evariste and Sophie both bought some stamps for their collections. Each stamp Evariste bought cost him £1.10, whilst Sophie paid 70p for each of her stamps. Between them they spend exactly £10. How many stamps did they buy in total?

A 9 B 10 C 11 D 12 E 13

Solution: **D**

Suppose that Evariste buys x stamps and Sophie buys y stamps. Then $1.10x + 0.70y = 10$ and hence $11x + 7y = 100$. We have one equation with two unknowns, so we need to use the fact that in this problem x and y are non-negative integers. As $y = \frac{100 - 11x}{7}$, we seek a value of x for which $100 - 11x$ is divisible by 7. Taking $x = 0, 1, 2, \ldots, 9$ we have $100 - 11x = 100, 89, 78, 67, 56, 45, 34, 23, 12, 1$. We see that only for $x = 4$ is $100 - 11x$ divisible by 7. So $x = 4$ and $y = \frac{56}{7} = 8$. So Sophie buys 8 stamps and Evariste buys 4 stamps, and they buy 12 stamps in total.

Extension Problem

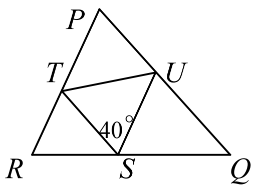
22.1 Solve this problem in the case where between them they spend exactly 71p.
22.2 Is it possible for Evariste and Sophie to spend exactly 70p between them?
22.3 What are the other possibilities for the total amount that Evariste and Sophie could have spent buying stamps costing £1.10 and 70p. In other words, for which values of N is it possible to find a solution of the equation

$$11x + 7y = N$$

where x and y are non-negative integers?

23. The points S, T, U lie on the sides of the triangle PQR, as shown, so that $QS = QU$ and $RS = RT$. $\angle TSU = 40^0$. What is $\angle TPU$?

A 60° B 70° C 80° D 90° E 100°



Solution: **E**

Let $\angle RST = x^\circ$ and $\angle QSU = y^\circ$. As $RS = RT$, the triangle RST is isosceles and hence $\angle RTS = RST = x^\circ$. Hence, as the angles in triangle RST add up to 180°, we have that $\angle SRT = (180 - 2x)^\circ$. Similarly $\angle SQU = (180 - 2y)^\circ$. Hence from triangle PQR, we have that $\angle TPU = (180 - (180 - 2x) - (180 - 2y))^\circ = (2(x + y) - 180)^\circ$. Now, as $\angle RST$, $\angle QSU$ and $\angle TSU$ are three angles on a straight line, $x + y + 40 = 180$. Therefore $x + y = 140$. We deduce that $\angle TPU = 2(140) - 180 = 100^\circ$. [Note that although we can find $x + y$, the information given does not enable us to calculate the values of x and y separately.]

24. Two adults and two children wish to cross a river. They make a raft but it will carry only the weight of one adult or two children. What is the minimum number of times the raft must cross the river to get all four people to the other side? (N.B. The raft may not cross the river without at least one person on board.)

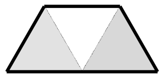
A 3 B 5 C 7 D 9 E 11

Solution: **D**

We note that if an adult crosses the river, there has to be a child on the other side to bring the raft back, otherwise the adult has to bring the raft back and two journeys have been wasted.

So the first journey must involve two of the children crossing, and one of them bringing the raft back. Then an adult can cross the river, and the child on the other bank can bring the raft back. In this way it has taken 4 journeys to get one adult across the river, with the other adult and the two children in their original position. With another 4 similar journeys, the second adult can cross the river. This leaves the two children in their original position. They can now cross the river in one journey. In this way it takes 9 journeys to get everyone across the river.

25. The diagram shows a trapezium made from three equilateral triangles. Three copies of the trapezium are placed together, without gaps or overlaps and so that complete edges coincide, to form a polygon with N sides.
How many different values of N are possible?



A 4 B 5 C 6 D 7 E 8

Solution: **C**

The three trapeziums have 12 edges in total. Whenever two of them are joined together, the total number of edges is reduced by at least 2. So the maximum possible value of N is $12 - 2 \times 2 = 8$. The minimum number of sides for a polygon is 3. So there are just 6 values for N, namely 3, 4, 5, 6, 7 and 8, that could possibly occur. The diagrams below show that they all can be achieved.