Intermediate Mathematical Challenge 2017

1. What is the value of $\frac{2}{5} + \frac{2}{50} + \frac{2}{500}$?

A 0.111 B 0.222 C 0.333 D 0.444 E 0.555

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1. D $\frac{2}{5} + \frac{2}{50} + \frac{2}{500} = \frac{200 + 20 + 2}{500} = \frac{222}{500} = \frac{444}{1000} = 0.444.$

2. Each of the diagrams below shows a circle and four small squares. In each case, the centre of the circle is the point where all four squares meet.
In one of the diagrams, exactly one third of the circle is shaded. Which one?

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2. **B** When one third of the circle is shaded, the angle at the centre of the shaded sector is $360° \div 3 = 120°$. In diagram A, the sector angle is 90°. In diagram C, the sector angle is $90° + 90° \div 2 = 135°$. So the correct diagram has a sector angle greater than that shown in A, but smaller than that shown in C. The only such sector angle is that in B.

3. How many squares have 7 as their units digit?

A 0 B 1 C 2 D 3 E 4

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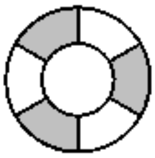
3. **A** Consider the units digits of the squares of the integers 0, 1, 2, 3, ..., 9. These are 0, 1, 4, 9, 6, 5, 6, 9, 4, 1. Note that none of these is 7, so no square ends in a 7.

4. Which of the following is *not* the sum of two primes?

A 5 B 7 C 9 D 11 E 13

4. D All primes except 2 are odd. So the sum of a pair of primes cannot be odd, and so cannot be prime, unless one of the pair is 2. We note that 5 = 2 + 3; 7 = 2 + 5; 9 = 2 +7; 13 = 2 + 11. However, 11 = 2 + 9 and so it is not the sum of two primes, as 9 is not prime.

5. The diagram shows two circles with the same centre. The radius of the outer circle is twice the radius of the inner circle. The region between the inner circle and the outer circle is divided into six equal segments as shown.
What fraction of the area of the outer circle is shaded?



A $\frac{3}{7}$ B $\frac{3}{8}$ C $\frac{3}{9}$ D $\frac{3}{10}$ E $\frac{3}{11}$

5. **B** Let the area of the inner circle be A. The radius of the outer circle is twice that of the inner circle, so its area is four times that of the inner circle , that is $4A$. Therefore the region between the two circles has area $3A$. As three of the six equal segments are shaded, the total shaded area is $3A \div 2$. So the required fraction is $\frac{3}{2}A \div 4A = \frac{3}{8}$.

6. The angles of a quadrilateral are in the ratio 3 : 4 : 5 : 6.

What is the difference between the largest angle and the smallest angle?

A 30° B 40° C 50° D 60° E 70°

6. **D** The sum of the interior angles of a quadrilateral is 360°. So the smallest angle in the quadrilateral is $3 \times \frac{360°}{3+4+5+6} = 60°$ and the largest angle is $6 \times \frac{360°}{3+4+5+6} = 120°$. Hence the required difference is 60°.

7. Four different positive integers are to be chosen so that they have a mean of 2017. What is the smallest possible range of the chosen integers?

A 2 B 3 C 4 D 5 E 6

7. C The smallest possible range of four positive integers is 3. Let these integers be n, $n+1, n+2, n+3$. For the mean of these four integers to equal 2017, we have $\frac{n+n+1+n+2+n+3}{4} = 2017$, that is $n = 2015\frac{1}{2}$. So the smallest possible range is not 3. However, note that the integers 2015, 2016, 2018 and 2019 have mean 2017 and range 4. So the smallest possible range of the integers is 4.

8. Which of the following numbers is the largest?

A 1.3542 B $1.354\dot{2}$ C $1.35\dot{4}\dot{2}$ D $1.3\dot{5}4\dot{2}$ E $1.\dot{3}54\dot{2}$

8. D Consider options B to E: $1.354\dot{2} = 1.354222\ldots$; $1.35\dot{4}\dot{2} = 1.35424242\ldots$; $1.3\dot{5}4\dot{2} = 1.3542542\ldots$; $1.\dot{3}54\dot{2} = 1.35423542\ldots$. These are all greater than 1.3542, so option A is not correct. The other four options have the same units digit and the first four digits after the decimal point are the same in all. So the largest option is that which has the largest digit five places after the decimal point and this is $1.3\dot{5}4\dot{2}$.

9. The number 'tu' is the two-digit number with units digit u and tens digit t. The digits a and b are distinct, and non-zero. What is the largest possible value of 'ab' $-$ 'ba' ?

A 81 B 72 C 63 D 54 E 45

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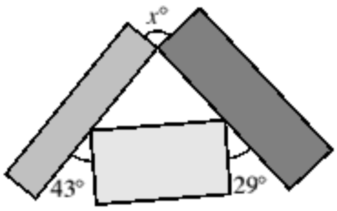
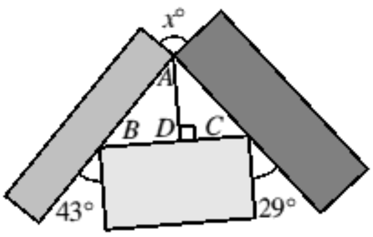
9. B First note that as 'ab' and 'ba' are two digit numbers, neither a nor b is equal to zero. Now 'ab' $-$ 'ba' $= (10a + b) - (10b + a) = 9a - 9b = 9(a - b)$. So for the difference to be as large as possible, a must be as large as possible, that is 9, and b must be as small as possible, that is 1.
So the required difference is $91 - 19 = 72$.

10. The diagram shows three rectangles.
What is the value of x?

A 108 B 104 C 100 D 96 E 92

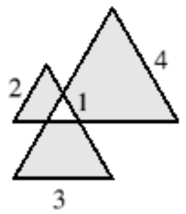


10. A As shown in the diagram, the perpendicular from A to BC meets BC at D. Because AD is parallel to two sides of the lower rectangle, $\angle BAD = 43°$ and $\angle DAC = 29°$ (corresponding angles in both cases). The angles at point A sum to $360°$, so $43 + 29 + 90 + x + 90 = 360$ and hence $x = 108$.

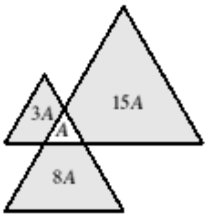


11. The diagram shows four equilateral triangles with sides of lengths 1, 2, 3 and 4. The area of the shaded region is equal to n times the area of the unshaded triangle of side-length 1. What is the value of n?

A 8 B 11 C 18 D 23 E 26



11. E Let the area of the equilateral triangle of side 1 be A. If similar figures have sides which are in the ratio $k : 1$, then the ratio of their areas is $k^2 : 1$. So the areas of the triangles with sides 2, 3, 4 are $4A$, $9A$, $16A$ respectively.
So, as shown in the diagram, the total shaded area is $3A + 8A + 15A = 26A$. Therefore $n = 26$.

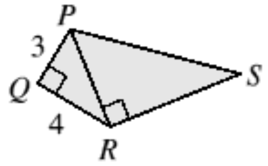


12. The combined age of Alice and Bob is 39. The combined age of Bob and Clare is 40. The combined age of Clare and Dan is 38. The combined age of Dan and Eve is 44. The total of all five ages is 105.
Which of the five is the youngest?

A Alice B Bob C Clare D Dan E Eve

12. D Let the ages of Alice, Bob, Clare, Dan and Eve be a, b, c, d, e respectively. So, for example, $b + c = 40$ and $d + e = 44$. Adding these gives $b + c + d + e = 84$. We are also told that $a + b + c + d + e = 105$. The difference between these equations shows that $a = 105 - 84 = 21$. Hence $b = 39 - 21 = 18$ and, similarly, $c = 40 - 18 = 22$, $d = 38 - 22 = 16$ and $e = 44 - 16 = 28$. So Dan is the youngest.

13. The diagram shows a quadrilateral *PQRS* made from two similar right-angled triangles, *PQR* and *PRS*. The length of *PQ* is 3, the length of *QR* is 4 and $\angle PRQ = \angle PSR$.
What is the perimeter of *PQRS*?



A 22 B $22\frac{5}{6}$ C 27 D 32 E $45\frac{1}{3}$

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13. A Pythagoras' Theorem shows that $PR = \sqrt{3^2 + 4^2} = 5$. So the perimeter of triangle *PQR* is 12. Since the triangles are similar and $PR : PQ = 5 : 3$ we see that the perimeter of triangle *PRS* is 20. Hence the perimeter of *PQRS* is $12 + 20 - 2 \times PR = 32 - 10 = 22$.

14. For what value of x is 64^x equal to 512^5 ?

A 6 B 7.5 C 8 D 16 E 40

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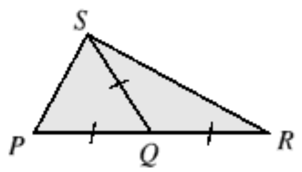
14. B Note that $64 = 2^6$ and $512 = 2^9$. Therefore $(2^6)^x = (2^9)^5$. So $2^{6x} = 2^{45}$. Hence $6x = 45$, that is $x = 7.5$.

15. In the diagram shown, $PQ = SQ = QR$ and $\angle SPQ = 2 \times \angle RSQ$.
What is the size of angle QRS?



A 20° B 25° C 30° D 35° E 40°

15. C Let $\angle QRS = x°$. Then, as $SQ = QR$, $\angle RSQ = x°$ also. Now $\angle SPQ = 2 \times \angle RSQ = 2x°$. In triangle PQS, $PQ = SQ$, so $\angle PSQ = \angle SPQ = 2x°$. Therefore $\angle PSR = \angle PSQ + \angle RSQ = 2x° + x° = 3x°$. The sum of the interior angles of a triangle is 180°. So, considering triangle PSR, $\angle SPR + \angle PSR + \angle PRS = 180°$. Therefore $2x° + 3x° + x° = 180°$. So $6x = 180$, that is $x = 30$.

16. The product of two positive integers is equal to twice their sum. This product is also equal to six times the difference between the two integers. What is the sum of these two integers?

A 6 B 7 C 8 D 9 E 10

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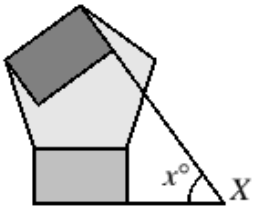
16. D Let the two positive integers be m and n. Then $mn = 2(m+n) = 6(m-n)$. So $2m + 2n = 6m - 6n$, that is $8n = 4m$. Therefore $m = 2n$. Substituting for m gives: $(2n)n = 2(2n + n)$. So $2n^2 = 6n$, that is $2n(n - 3) = 0$. Therefore $n = 0$ or 3. However, n is positive so the only solution is $n = 3$. Therefore $m = 2 \times 3 = 6$ and $m + n = 6 + 3 = 9$.

17. The diagram shows two rectangles and a regular pentagon. One side of each rectangle has been extended to meet at X. What is the value of x?



A 52 B 54 C 56 D 58 E 60

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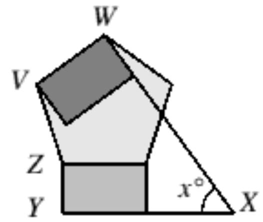
17. B Since $VWXYZ$ is a pentagon, the sum of its interior angles is 540°. Now $\angle ZVW$ is an interior angle of a regular pentagon and so is 108°. Both $\angle VWX$ and $\angle XYZ$ are 90°; and the reflex angle $\angle YZV = 90° + 108°$. Therefore $540° = 108° + 90° + 90° + 90° + 108° + x°$. Hence $x = 540 - 486 = 54$.



18. A water tank is $\frac{5}{6}$ full. When 30 litres of water are removed from the tank, the tank is $\frac{4}{5}$ full. How much water does the tank hold when full?

A 180 litres B 360 litres C 540 litres D 720 litres E 900 litres

18. E Let the capacity of the tank be x litres. Then $30 = \frac{5x}{6} - \frac{4x}{5} = \frac{25x - 24x}{30} = \frac{x}{30}$. So $x = 30 \times 30 = 900$.

19. $PQRS$ is a square. Point T lies on PQ so that $PT : TQ = 1:2$. Point U lies on SR so that $SU : UR = 1 : 2$. The perimeter of $PTUS$ is 40 cm.
What is the area of $PTUS$?

A 40 cm^2 B 45 cm^2 C 48 cm^2 D 60 cm^2 E 75 cm^2

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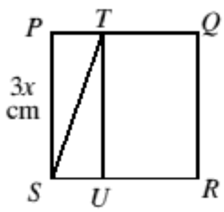
19. E Let square $PQRS$ have side $3x$ cm. Then, as $PT : TQ = 1 : 2$, $PT = x$ cm. Similarly, $US = x$ cm. In triangles PTS and UST, $PT = US$, $\angle PTS = \angle UST$ (alternate angles) and TS is common to both. So the triangles are congruent (SAS). Therefore $UT = PS = 3x$ cm and $\angle TUS = \angle SPT = 90°$. Hence $PTUS$ is a rectangle, which has perimeter 40 cm. So $40 = 2(3x + x) = 8x$. Therefore $x = 5$ and the area of $PTUS$, in cm^2, is $15 \times 5 = 75$.



20. The diagram shows seven circular arcs and a heptagon with equal sides but unequal angles. The sides of the heptagon have length 4. The centre of each arc is a vertex of the heptagon, and the ends of the arc are the midpoints of the two adjacent sides.
What is the total shaded area?

A 12π B 14π C 16π D 18π E 20π

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20. D First note that as the length of each side of the heptagon is 4, the radius of each of the seven arcs is 2. The sum of the interior angles of a heptagon is $(7 - 2) \times 180° = 900°$. So the sum of the angles subtended by the circular arcs at the centres of the circles of radius 2 cm is $7 \times 360° - 900° = \left(7 - \frac{5}{2}\right) \times 360° = \frac{9}{2} \times 360°$. Therefore the total shaded area is equal to the total area of $\frac{9}{2}$ circles of radius 2. So the total shaded area is $\frac{9}{2} \times \pi \times 2^2 = 18\pi$.

21. *Brachycephalus* frogs are tiny – less than 1 cm long – and have three toes on each foot and two fingers on each 'hand', whereas the common frog has five toes on each foot and four fingers on each 'hand'.

Some *Brachycephalus* and common frogs are in a bucket. Each frog has all its fingers and toes. Between them they have 122 toes and 92 fingers.

How many frogs are in the bucket?

A 15 B 17 C 19 D 21 E 23

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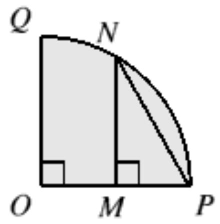
21. A Let the number of Brachycephalus frogs and common frogs in the bucket be b and c respectively. Note that each Brachycephalus frog has 6 toes and 4 fingers, while a common frog has 10 toes and 8 fingers.

Therefore, $6b + 10c = 122$ (1); $4b + 8c = 92$ (2). Subtracting (2) from (1) gives $2b + 2c = 30$, so $b + c = 15$.

22. The diagram shows an arc PQ of a circle with centre O and radius 8. Angle QOP is a right angle, the point M is the midpoint of OP and N lies on the arc PQ so that MN is perpendicular to OP.

Which of the following is closest to the length of the perimeter of triangle PNM ?



A 17 B 18 C 19 D 20 E 21

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22. C Consider triangles MON and MPN. Note that $MO = MP = 4$ because M is the midpoint of OP; $\angle OMN = \angle PMN = 90°$ because MN is perpendicular to OP; side NM is common to both triangles. So triangles MON and MPN are congruent (SAS). Therefore $PN = ON = 8$ because ON is a radius of the circle. By Pythagoras' Theorem in triangle OMN, $ON^2 = OM^2 + MN^2$, so $8^2 = 4^2 + MN^2$. Therefore $MN^2 = 8^2 - 4^2 = 48$. So $MN = \sqrt{48}$.
The perimeter of triangle PNM is $PN + NM + MP = 8 + \sqrt{48} + 4 = 12 + \sqrt{48}$.
Now $6.5 < \sqrt{48} < 7$, since $\sqrt{42.25} < \sqrt{48} < \sqrt{49}$. So $\sqrt{48}$ is closer in value to 7 than it is to 6. So $12 + \sqrt{48}$ is nearer to 19 than it is to 18.

23. Two brothers and three sisters form a single line for a photograph. The two boys refuse to stand next to each other.
How many different line-ups are possible?

A 24 B 36 C 60 D 72 E 120

23. D Let the five positions in the photograph be numbered 1, 2, 3, 4, 5. Then the boys may occupy a total of six positions: 1 and 3; 1 and 4; 1 and 5; 2 and 4; 2 and 5; 3 and 5. For each of these positions, the boys may be arranged in two ways as they can interchange places. So there are 12 ways of positioning the boys. For each of these, the girls must be placed in three positions. In each case, the first girl may choose any one of three positions, the second girl may choose either of two positions and then there is just one place remaining for the third girl. So for each arrangement of the two boys there are $3 \times 2 \times 1$ different ways of arranging the three girls. Therefore the total number of line-ups is $12 \times 6 = 72$.

24. The nth term in a certain sequence is calculated by multiplying together all the numbers $\sqrt{1 + \frac{1}{k}}$, where k takes all the integer values from 2 to $n + 1$ inclusive. For example, the third term in the sequence is $\sqrt{1 + \frac{1}{2}} \times \sqrt{1 + \frac{1}{3}} \times \sqrt{1 + \frac{1}{4}}$.

Which is the smallest value of n for which the nth term of the sequence is an integer?

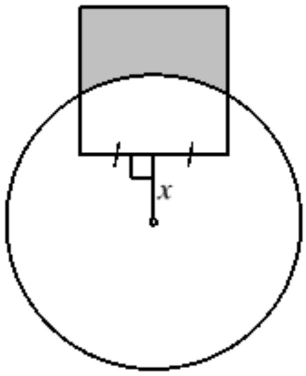
A 3 B 5 C 6 D 7 E more than 7

1734

24. C The product may be written $\sqrt{\frac{3}{2}} \times \sqrt{\frac{4}{3}} \times \sqrt{\frac{5}{4}} \times \sqrt{\frac{6}{5}} \times \ldots = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{4}}{\sqrt{3}} \times \frac{\sqrt{5}}{\sqrt{4}} \times \frac{\sqrt{6}}{\sqrt{5}} \times \ldots$.
Notice that the numerator of each fraction is cancelled out by the denominator of the following fraction and the only terms which are not cancelled are the denominator of the first fraction and the numerator of the last fraction. So the nth term of the sequence is $\frac{\sqrt{n+2}}{\sqrt{2}} = \sqrt{\frac{n+2}{2}}$. As $n \geqslant 1$ the product is not equal to 1.
The product increases with n. So the next possible integer value to consider is 2 and this does occur when $n = 6$ as $\sqrt{\frac{6+2}{2}} = \sqrt{4} = 2$.
So the smallest number of terms required for the product to be an integer is 6.

25. The diagram shows a circle with radius 2 and a square with sides of length 2. The centre of the circle lies on the perpendicular bisector of a side of the square, at a distance x from the side, as shown. The shaded region – inside the square but outside the circle – has area 2.

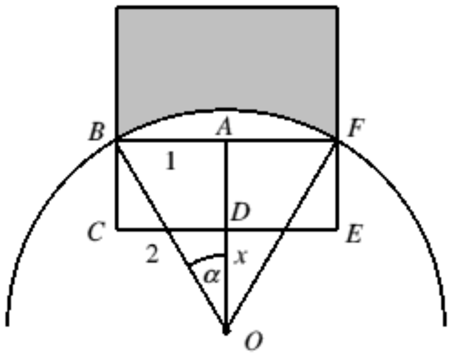
What is the value of x?



A $\frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$ B $\frac{\pi}{3} + \frac{\sqrt{3}}{4} - 1$ C $\frac{\pi}{3} + \frac{1}{2}$ D $\frac{\pi}{3} + 1$ E $\frac{\pi}{3}$

1735

25. A The diagram shows part of the diagram in the question. Point O is the centre of the circle and points A, B, C, D, E, F, are as shown. Consider triangle BOF: BF is equal in length to the side of the square so $BF = 2$. Also $OB = OF = 2$ as they are both radii of the circle. So triangle BOF is equilateral. Therefore $\angle BOF = 60°$, so the area of sector $OBF = \frac{60}{360} \times \pi \times 2^2 = \frac{2\pi}{3}$.



By Pythagoras' Theorem: $OA = \sqrt{2^2 - 1^2} = \sqrt{3}$.
So the area of triangle $OBF = \frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$. Therefore the area of the segment bounded by arc BF and line segment $BF = \frac{2\pi}{3} - \sqrt{3}$. The area of rectangle $BFEC = BF \times AD = 2 \times (\sqrt{3} - x)$.

The shaded region has area 2, so the area of the above segment + area of rectangle $BFEC$ = area of the given square minus 2 = $4 - 2 = 2$.

Hence $\frac{2\pi}{3} - \sqrt{3} + 2\sqrt{3} - 2x = 2$. So $x = \frac{\pi}{3} + \frac{\sqrt{3}}{2} - 1$.