Intermediate Mathematical Challenge 2016

1. What is the value of $6102 - 2016$?

A 3994 B 4086 C 4096 D 4114 E 4994

1611

1. B $6116 - 2016 = 4100$, so $6102 - 2016 = 4100 - 14 = 4086$.

2. Which of the following fractions is closest to 1?

A $\frac{7}{8}$ B $\frac{8}{7}$ C $\frac{9}{10}$ D $\frac{10}{11}$ E $\frac{11}{10}$

1612

2. D The difference between the given options and 1 is $\frac{1}{8}$, $\frac{1}{7}$, $\frac{1}{10}$, $\frac{1}{11}$ and $\frac{1}{10}$ respectively. As $\frac{1}{11}$ is the smallest of these fractions, $\frac{10}{11}$ is closest to 1.

3. How many of these five expressions give answers which are *not* prime numbers?

$1^2 + 2^2$ $2^2 + 3^2$ $3^2 + 4^2$ $4^2 + 5^2$ $5^2 + 6^2$

A 0 B 1 C 2 D 3 E 4

3. B The values of the five expressions are 5, 13, 25, 41 and 61 respectively. Of these, only 25 is non-prime.

4. Amrita is baking a cake today. She bakes a cake every fifth day. How many days will it be before she next bakes a cake on a Thursday?

A 5 B 7 C 14 D 25 E 35

1614

4. E Amrita bakes every 5 days and Thursdays come every 7 days. So the next time Amrita bakes on a Thursday will be in 35 days time since 35 is the lowest common multiple of 5 and 7.

5. When travelling from London to Edinburgh by train, you pass a sign saying ‘Edinburgh 200 miles’. Then, $3\frac{1}{2}$ miles later, you pass another sign saying ‘Half way between London and Edinburgh’.

How many miles is it by train from London to Edinburgh?

A 393 B $396\frac{1}{2}$ C 400 D $403\frac{1}{2}$ E 407

1615

5. **A** By train, the distance in miles of the second sign from Edinburgh is $200 - 3\frac{1}{2}$. This sign is halfway between London and Edinburgh, so the distance in miles between the two cities is $2(200 - 3\frac{1}{2}) = 400 - 7 = 393$.

6. One third of the animals in Jacob's flock are goats, the rest are sheep. There are twelve more sheep than goats.

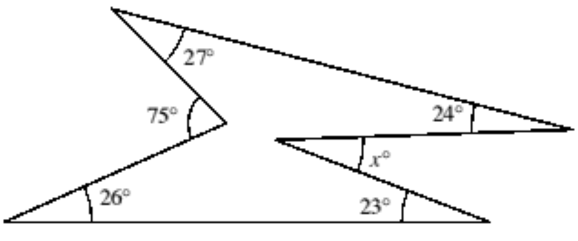
How many animals are there altogether in Jacob's flock?

A 12 B 24 C 36 D 48 E 60

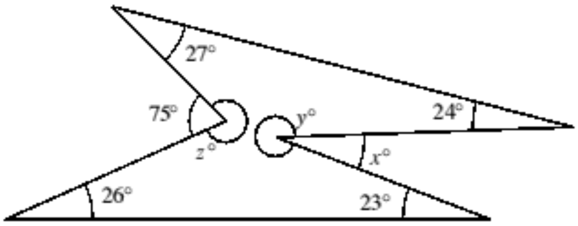
6. **C** Let g and s be the number of goats and sheep respectively. Then $s = 2g$ and $12 = s - g = 2g - g = g$. Hence the number of animals is $s + g = 3g = 36$.

7. In the diagram, what is the value of x?

A 23 B 24 C 25 D 26 E 27



7. **C** The angles at a point sum to 360° so $75 + z = 360$ and $y + x = 360$. Therefore $75 + z + y + x = 720$.



The sum of the interior angles of a hexagon is $4 \times 180° = 720°$.
Therefore
$27 + 24 + y + 23 + 26 + z = 720$, so $75 + z + y + x = 27 + 24 + y + 23 + 26 + z$.
Hence $75 + x = 27 + 24 + 23 + 26 = 100$. So $x = 100 - 75 = 25$.

8. What is the value of $2.017 \times 2016 - 10.16 \times 201.7$?

A 2.016 B 2.017 C 20.16 D 2016 E 2017

8. **E** $2.017 \times 2016 - 10.16 \times 201.7 = 201.7 \times 20.16 - 10.16 \times 201.7$
$= 201.7(20.16 - 10.16) = 201.7 \times 10 = 2017$.

9. The world's fastest tortoise is acknowledged to be a leopard tortoise from County Durham called Bertie. In July 2014, Bertie sprinted along a 5.5 m long track in an astonishing 19.6 seconds.

What was Bertie's approximate average speed in km per hour?

A 0.1 B 0.5 C 1 D 5 E 10

1619

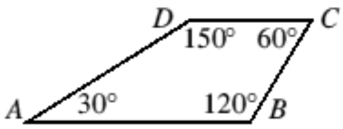
9. C Bertie travelled 5.5 m in 19.6 s, which is just less than one-third of a minute. So his average speed was approximately 16.5 m per minute, which is equal to 990 m in one hour, as $16.5 \times 60 = 990$. Now 990 m = 0.99 km, so Bertie's approximate average speed was 1 km per hour.

10. The angles of a quadrilateral taken in order are $x°$, $5x°$, $2x°$ and $4x°$. Which of the following is the quadrilateral?

A kite B parallelogram C rhombus D arrowhead E trapezium

1620

10. E The sum of the interior angles of a quadrilateral is $360°$, so $x + 5x + 2x + 4x = 360$, that is $12x = 360$. Therefore $x = 30$ and the angles of the quadrilateral, taken in order, are $30°$, $150°$, $60°$ and $120°$. The diagram shows the shape of the quadrilateral.

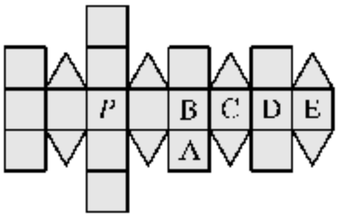


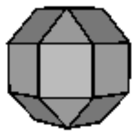
Since $30 + 150 = 180$, we see that AB and DC are parallel. Since it has no equal angles, it is not a rhombus or a parallelogram so it is a trapezium.

11. The net shown consists of squares and equilateral triangles. The net is folded to form a rhombicuboctahedron, as shown.

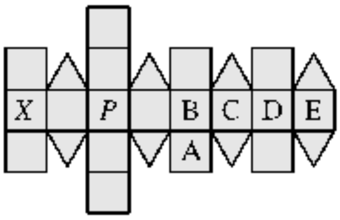
When the face marked P is placed face down on a table, which face will be facing up?

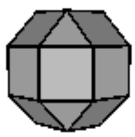
A B C D E





11. D When the net is folded up to form the rhombicuboctahedron, the left-hand edge of the square marked X is joined to the right-hand edge of the square marked E so that the eight squares at the centre of the net form a band around the solid. In this band, the square opposite square P is the square which is four squares away from P, that is square D. So if the square marked P is placed face down on a table, then the square marked D will be facing up.





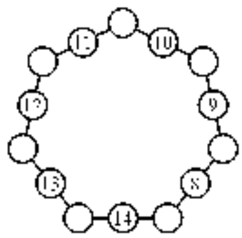
12. The sum of two numbers a and b is 7 and the difference between them is 2.
What is the value of $a \times b$?

A $8\frac{1}{4}$ B $9\frac{1}{4}$ C $10\frac{1}{4}$ D $11\frac{1}{4}$ E $12\frac{1}{4}$

1622

12. D Assume that $a > b$. Then $a + b = 7$ and $a - b = 2$. Adding these two equations together gives $2a = 9$. So $a = \frac{9}{2}$ and hence $b = 7 - \frac{9}{2} = \frac{14-9}{2} = \frac{5}{2}$. Therefore $a \times b = \frac{9}{2} \times \frac{5}{2} = \frac{45}{4} = 11\frac{1}{4}$.

13. The diagram shows a heptagon with a line of three circles on each side. Each circle is to contain exactly one number. The numbers 8 to 14 are distributed as shown and the numbers 1 to 7 are to be distributed to the remaining circles. The total of the numbers in each of the lines of three circles is to be the same.
What is this total?



A 18 B 19 C 20 D 21 E 22

1623

13. B In the seven lines each of the integers from 1 to 7 is used twice and each of the integers from 8 to 14 is used once. So the sum of the numbers in the seven lines is $(1 + 2 + \ldots +14) + (1 + 2 + \ldots +7) = 105 + 28 = 133$. Therefore the total of the numbers in each line is $133 \div 7 = 19$.

It is left as an exercise for the reader to show that it is possible to complete the diagram so that the total of the three numbers in each line is indeed 19.

14. Tegwen has the same number of brothers as she has sisters. Each one of her brothers has 50% more sisters than brothers.

How many children are in Tegwen's family?

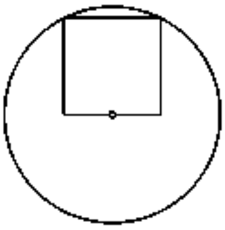
A 5 B 7 C 9 D 11 E 13

14. D Let there be g girls and b boys in Tegwen's family. Then, as she has the same number of brothers as she does sisters, $b = g - 1$. Also, each of her brothers has 50% more sisters than brothers. Therefore $g = \frac{3}{2}(b - 1)$. So $b + 1 = \frac{3}{2}(b - 1)$ and hence $2b + 2 = 3b - 3$. Rearranging this equation gives $b = 5$. So $g = 5 + 1 = 6$. Therefore there are $5 + 6 = 11$ children in Tegwen's family.

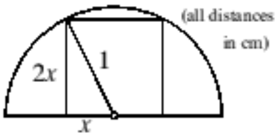
15. The circle has radius 1 cm. Two vertices of the square lie on the circle. One edge of the square goes through the centre of the circle, as shown.

What is the area of the square?

A $\frac{4}{5}$ cm^2 B $\frac{\pi}{5}$ cm^2 C 1 cm^2 D $\frac{\pi}{4}$ cm^2 E $\frac{5}{4}$ cm^2



15. A Let the length of the side of the square be $2x$ cm. Then, using Pythagoras' Theorem in the triangle shown, $(2x)^2 + x^2 = 1^2$. So $4x^2 + x^2 = 1$. Therefore $x^2 = \frac{1}{5}$ and the area of the square is $4x^2$ cm^2 $= \frac{4}{5}$ cm^2.



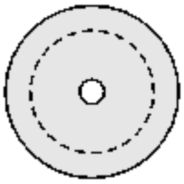
16. How many of the following positive integers are divisible by 24?

$2^2 \times 3^2 \times 5^2 \times 7^3$ $2^2 \times 3^2 \times 5^3 \times 7^2$ $2^2 \times 3^3 \times 5^2 \times 7^2$ $2^3 \times 3^2 \times 5^2 \times 7^2$

A 0 B 1 C 2 D 3 E 4

16. B The prime factorisation of $24 = 2^3 \times 3$. Therefore all multiples of 24 must include both 2^3 and 3 in their prime factorisation. Of the options given, only the last includes 2^3. As it is also a multiple of 3, it is a multiple of 24.

17. The shaded region in the diagram, bounded by two concentric circles, is called an *annulus*. The circles have radii 2 cm and 14 cm.



The dashed circle divides the area of this annulus into two equal areas.

What is its radius?

A 9 cm B 10 cm C 11 cm D 12 cm E 13 cm

1627 ©UKMT

17. B Let the radius of the dashed circle be r cm. Then one of the equal areas is bounded by circles of radii of 14 cm and r cm, whilst the other is bounded by circles of radii of r cm and 2 cm. So $\pi \times 14^2 - \pi r^2 = \pi r^2 - \pi \times 2^2$. Dividing throughout by π gives $196 - r^2 = r^2 - 4$. So $2r^2 = 200$, that is $r^2 = 100$. Therefore $r = 10$ (since $r > 0$).

18. The sum of the areas of the squares on the sides of a right-angled isosceles triangle is 72 cm^2.
What is the area of the triangle?

A 6 cm^2 B 8 cm^2 C 9 cm^2 D 12 cm^2 E 18 cm^2

1628 ©UKMT

18. C Let the length of each of the shorter sides of the triangle be x cm and the length of its hypotenuse be y cm. Then, by Pythagoras' Theorem: $x^2 + x^2 = y^2$. So $y^2 = 2x^2$. Also, $x^2 + x^2 + y^2 = 72$, so $4x^2 = 72$, that is $x^2 = 18$. Now the area of the triangle, in cm^2, is $\frac{1}{2} \times x \times x = \frac{1}{2}x^2 = 9$.

19. A list of positive integers has a median of 8, a mode of 9 and a mean of 10.
What is the smallest possible number of integers in the list?

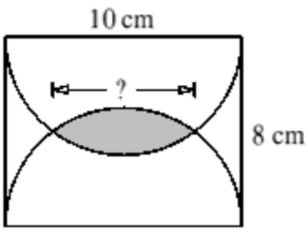
A 5 B 6 C 7 D 8 E 9

1629 ©UKMT

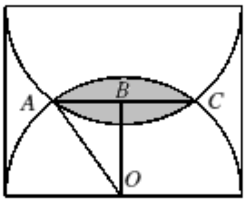
19. B From the information given, there are at least two 9s in the list, since 9 is the mode, and at least one number greater than 10, since 10 is the mean. So there are at least three numbers greater than 8 in the list. Therefore the list must contain at least six numbers, as the median of the numbers is 8. Moreover, it is possible to find suitable lists of six numbers with sum 60 (as the mean is 10), for example 1, 2, 7, 9, 9, 32.

20. Two semicircles are drawn in a rectangle as shown.
What is the width of the overlap of the two semicircles?

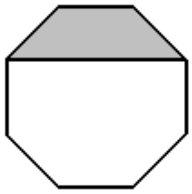
A 3 cm B 4 cm C 5 cm D 6 cm E 7 cm



20. D In the diagram, O is the centre of the lower semicircle, A and C are the points of intersection of the two semicircles and B is the point at the centre of the rectangle and also of the overlap. Now OA is a radius of the semicircle so OA has length 5 cm. Also OB is half the height of the rectangle so has length 4 cm. Angle ABO is a right angle. So triangle ABO is a (3, 4, 5) triangle and hence $AC = 2 \times 3\text{ cm} = 6\text{ cm}$.

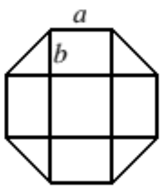


21. The diagram shows a regular octagon. What is the ratio of the area of the shaded trapezium to the area of the whole octagon?



A $1:4$ B $5:16$ C $1:3$ D $\sqrt{2}:2$ E $3:8$

21. A Let a be the side length of the octagon and b be as shown on the diagram. The square in the centre is a by a, each rectangle is a by b and the triangles are each half of a b by b square. Applying Pythagoras' Theorem to a triangle shows that $a^2 = 2b^2$. So the shaded area is $b^2 + ab = b^2 + \sqrt{2}b^2 = b^2(1 + \sqrt{2})$.



Similarly the total area of the figure is $a^2 + 4ab + 2b^2 = 4b^2 + 4\sqrt{2}b^2 = b^2(4 + 4\sqrt{2})$. Therefore the ratio required is $(1 + \sqrt{2}) : (4 + 4\sqrt{2}) = 1 : 4$.

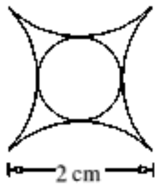
22. In a particular group of people, some always tell the truth, the rest always lie. There are 2016 in the group. One day, the group is sitting in a circle. Each person in the group says, "Both the person on my left and the person on my right are liars."

What is the difference between the largest and smallest number of people who could be telling the truth?

A 0 B 72 C 126 D 288 E 336

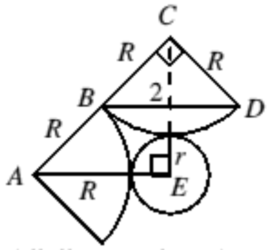
22. E For brevity, let T denote a truth teller and L a liar. Clearly each T has to have an L on each side. Each L either (i) has a T on each side or (ii) has an L on one side and a T on the other side. The largest number of Ts will occur if (i) is always the case. This gives the arrangement TLTLTL… which, since 2 divides 2016, joins up correctly after going round the table. In this case the number of Ts is $\frac{1}{2} \times 2016$. The smallest will occur if case (ii) always is the case. This gives the arrangement LLTLLTLLT… which, since 3 divides 2016, also joins up correctly. In this case the number of Ts is $\frac{1}{3} \times 2016$. The difference is $\frac{1}{6} \times 2016 = 336$.

23. A Saxon silver penny, from the reign of Ethelbert II in the eighth century, was sold in 2014 for £78 000. A design on the coin depicts a circle surrounded by four equal arcs, each a quarter of a circle, as shown. The width of the design is 2 cm. What is the radius of the small circle, in centimetres?



A $\frac{1}{2}$ B $2-\sqrt{2}$ C $\frac{1}{2}\sqrt{2}$ D $5-3\sqrt{2}$ E $2\sqrt{2}-2$

23. B The diagram shows part of the figure, to which have been added A and C, centres of two of the quarter-circle arcs, B and D, points of intersection of two arcs, and E, the centre of the small circle. In cm, the radii of each arc and the small circle are R and r respectively. Firstly, note that $\angle BCD$ is a right angle as arc BD is a quarter of a circle. Therefore, by Pythagoras' Theorem $R^2 + R^2 = 2^2$ so $R = \sqrt{2}$. Consider triangle ACE: from the symmetry of the figure we deduce that $\angle AEC = \frac{1}{4} \times 360° = 90°$. So, by Pythagoras' Theorem $(R+r)^2 + (R+r)^2 = (2R)^2 = 4R^2$. Therefore $(R + r)^2 = 2R^2 = 2 \times 2 = 4$. Hence $R + r = 2$, so $r = 2 - R = 2 - \sqrt{2}$.



(all distances in cm)

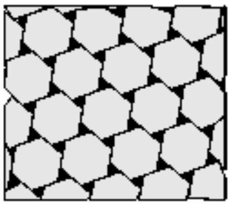
24. Every day, Aimee goes up an escalator on her journey to work. If she stands still, it takes her 60 seconds to travel from the bottom to the top. One day the escalator was broken so she had to walk up it. This took her 90 seconds.

How many seconds would it take her to travel up the escalator if she walked up at the same speed as before while it was working?

A 30 B 32 C 36 D 45 E 75

24. C Let the distance from the bottom of the escalator to the top be d. Then, when she stands still, Aimee travels $d/60$ every second. When she is walking, Aimee travels $d/90$ every second. So when Aimee walks up the working escalator, the distance which she travels every second is $\frac{d}{60}+\frac{d}{90}=\frac{3d+2d}{180}=\frac{5d}{180}=\frac{d}{36}$. So the required number of seconds is 36.

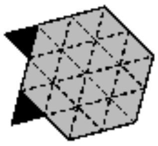
25. The tiling pattern shown uses two types of tile, regular hexagons and equilateral triangles, with the length of each side of the equilateral triangles equal to half the length of each side of the hexagons. A large number of tiles is used to cover a floor.



Which of the following is closest to the fraction of the floor that is shaded black?

A $\frac{1}{8}$ B $\frac{1}{10}$ C $\frac{1}{12}$ D $\frac{1}{13}$ E $\frac{1}{16}$

25. D The tiled area may be considered to be a tessellation of the figure shown, except for the dotted lines. For every hexagonal tile, there are two triangular tiles. The diagram shows that the area of each hexagonal tile is 24 times the area of each triangular tile. As there are two triangular tiles to each hexagonal tile, the ratio of the fraction of the floor shaded black to that which is shaded grey is 2 : 24 = 1: 12. Therefore, in the repeating pattern of tiles, the fraction which is shaded black is 1/13.



The exact ratios given are for the infinite plane. Since we are dealing with a finite floor, this is approximate since the edges are unpredictable, but close to correct since the numbers involved are large.