Intermediate Mathematical Challenge 2014

1. What is 25 % of $\frac{3}{4}$?

A $\frac{3}{16}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D 1 E 3

1411 ©UKMT

1. A 25% of $\frac{3}{4} = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$.

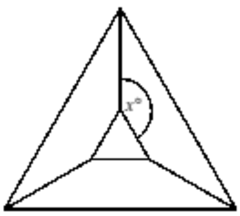
2. Which is the smallest positive integer for which all these are true?
   (i) It is odd.
   (ii) It is not prime.
   (iii) The next largest odd integer is not prime.

   A 9 B 15 C 21 D 25 E 33

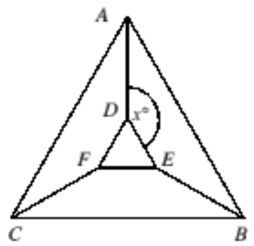
2. D The first four options are the smallest positive integers which are both odd and not prime. However, the next largest odd numbers after 9, 15, 21 are 11, 17, 23 respectively and these are all prime. The next largest odd number after 25 is 27, which is not prime. So 25 is the smallest positive integer which satisfies all three conditions.

3. An equilateral triangle is placed inside a larger equilateral triangle so that the diagram has three lines of symmetry. What is the value of x?

   A 100 B 110 C 120 D 130 E 150



3. **E** Clearly AD lies along one of the lines of symmetry of the figure. So $\angle FDA = \angle EDA = x°$. Triangle DEF is equilateral so $\angle EDF = 60°$.
The angles which meet at a point sum to 360°, so
$x + x + 60 = 360$.
Therefore $x = 150$.



4. You are given that m is an even integer and n is an odd integer. Which of these is an odd integer ?

A $3m + 4n$ B $5mn$ C $(m + 3n)^2$ D m^3n^3 E $5m + 6n$

4. **C** Since m is even, $m = 2k$ for some integer k. So $3m + 4n = 2(3k + 2n)$; $5mn = 2(5kn)$; $m^3n^3 = 8k^3n^3$ and $5m + 6n = 2(5k + 3n)$, which are all even. As n is odd, $3n$ is also odd. So $m + 3n$ is an even integer plus an odd integer and is therefore odd. The square of an odd integer is odd so $(m + 3n)^2$ is odd.

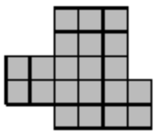
5. A ship's bell is struck every half hour, starting with one bell at 0030, two bells (meaning the bell is struck twice) at 0100, three bells at 0130 until the cycle is complete with eight bells at 0400. The cycle then starts again with one bell at 0430, two bells at 0500 and so on. What is the total number of times the bell is struck between 0015 on one day and 0015 on the following day?

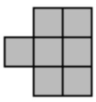
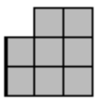
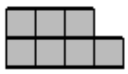
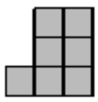
A 24 B 48 C 108 D 144 E 216

1415

5. E In one complete cycle of 4 hours, the clock is struck $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ times. So in 24 hours the clock is struck $6 \times 36 = 216$ times.

6. The shape shown on the right was assembled from three identical copies of one of the smaller shapes below, without gaps or overlaps.
Which smaller shape was used?



A  B  C  D  E 

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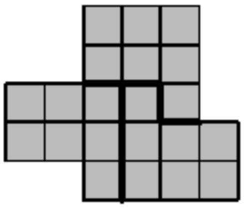
6. **E** The large shape consists of 21 small squares, so the required shape is made up of 7 small squares. So A and C may be eliminated. The diagram on the right shows that shape E is as required. It is left to the reader to check that neither B nor D was the shape used.



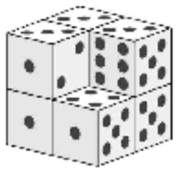
7. Just one positive integer has exactly 8 factors including 6 and 15. What is the integer?

A 21 B 30 C 45 D 60 E 90

7. **B** Since 6 and 15 are factors of the integer, its prime factors will include 2, 3 and 5. So 10 and 30 will also be factors of the required integer. Seven of its factors are now known and as 1 must also be a factor, the required integer is 30, the factors of which are 1, 2, 3, 5, 6, 10, 15, 30.

(*Positive integers with exactly 8 factors are of the form* pqr *or* pq^3 *or* p^7 *where* p, q, r *are distinct primes.*)

8. A large cube is made by stacking eight dice. The diagram shows the result, except that one of the dice is missing. Each die has faces with 1, 2, 3, 4, 5 and 6 pips and the total number of pips on each pair of opposite faces is 7. When two dice are placed face to face, the matching faces must have the same number of pips. What could the missing die look like?



A  B  C  D  E 

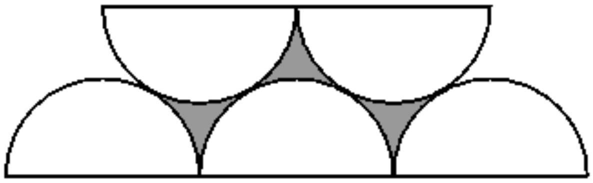
8. C The missing die, if correctly placed in the figure, would show faces 1, 3, 5 placed in a clockwise direction around the nearest corner. An examination of each of the five proposed dice shows that only C has this property.

9. At the age of twenty-six, Gill has passed her driving test and bought a car. Her car uses p litres of petrol per 100 km travelled. How many litres of petrol would be required for a journey of d km?

A $\frac{pd}{100}$ B $\frac{100p}{d}$ C $\frac{100d}{p}$ D $\frac{100}{pd}$ E $\frac{p}{100d}$

9. A Gill's car uses $p / 100$ litres of petrol for every one kilometre travelled. So for a journey of length d km, $pd / 100$ litres of petrol are required.

10. The diagram shows five touching semicircles, each with radius 2.



What is the length of the perimeter of the shaded shape?

A 5π B 6π C 7π D 8π E 9π

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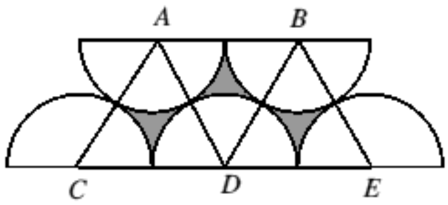
10. B A, B, C, D, E are the centres of the five semicircles. Note that AC joins the centres of two touching semicircles and therefore passes through the point of contact of the semicircles. So AC has length $2 + 2 = 4$. This also applies to all of the other sides of triangles ACD and BED. Hence both triangles are equilateral. So each of the nine arcs which make up the perimeter of the shaded shape subtends an angle of 60° at the centre of a semicircle.



11. Not all characters in the Woodentops series tell the truth. When Mr Plod asked them, "How many people are there in the Woodentops family?", four of them replied as follows:

Jenny: "An even number." Willie: "An odd number." Sam: "A prime number."

Mrs Scrubitt: "A number which is the product of two integers greater than one."

How many of these four were telling the truth?

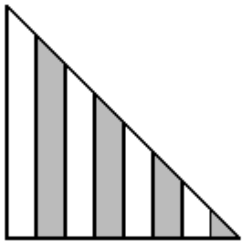
A 0 B 1 C 2 D 3 E 4

1421

11. C Precisely one of Jenny and Willie is telling the truth since the number of people is either even or odd. Similarly, precisely one of Sam and Mrs Scrubitt is telling the truth since the number of people is either a prime number or a number which is the product of two integers greater than one. So although it is not possible to deduce who is telling the truth, it is possible to deduce that exactly two of them are doing so.

12. The diagram shows an isosceles right-angled triangle divided into strips of equal width. Four of the strips are shaded. What fraction of the area of the triangle is shaded?

A $\frac{11}{32}$ B $\frac{3}{8}$ C $\frac{13}{32}$ D $\frac{7}{16}$ E $\frac{15}{32}$



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12. D Let the width of each strip be 1 unit. Then the triangle has base 8 and perpendicular height 8. So its area is equal to $\frac{1}{2} \times 8 \times 8 = 32$. Looking from the right, the area of the first shaded strip is 1 unit of area less than the first unshaded strip. This difference of 1 unit also applies to the other three pairs of strips in the triangle, which means that the shaded area is 4 less than the unshaded area. So the total shaded area is $\frac{1}{2}(32 - 4) = 14$. Therefore the required fraction is $\frac{14}{32} = \frac{7}{16}$.

13. How many numbers can be written as a sum of two different positive integers each at most 100?

A 100 B 197 C 198 D 199 E 200

13. B The smallest such number is $1 + 2 = 3$, whilst the largest is $99 + 100 = 199$. Every number between 3 and 199 may be written as $1 + n$ with $n = 2, 3, \ldots, 99$ or as $100 + n$ with $n = 1, \ldots, 99$. So in total there are $(199 - 3) + 1 = 197$ such numbers.

14. This year the *Tour de France* starts in Leeds on 5 July. Last year, the total length of the *Tour* was 3404 km and the winner, Chris Froome, took a total time of 83 hours 56 minutes 40 seconds to cover this distance. Which of these is closest to his average speed over the whole event?

A 32 km/h B 40 km/h C 48 km/h D 56 km/h E 64 km/h

14. B Chris Froome's average speed $\approx \frac{3400}{84}$ km/h $\approx \frac{3400}{85}$ km/h $= \frac{200}{5}$ km/h $= 40$ km/h.

15. Zac halves a certain number and then adds 8 to the result. He finds that he obtains the same answer if he doubles his original number and then subtracts 8 from the result.
What is Zac's original number?

A $8\frac{2}{3}$ B $9\frac{1}{3}$ C $9\frac{2}{3}$ D $10\frac{1}{3}$ E $10\frac{2}{3}$

15. E Let Zac's number be x. Then $\frac{1}{2}x + 8 = 2x - 8$. So $x + 16 = 4x - 16$. Therefore $32 = 3x$, that is $x = 10\frac{2}{3}$.

16. The base of a triangle is increased by 25% but the area of the triangle is unchanged. By what percentage is the corresponding perpendicular height decreased?

A $12\frac{1}{2}\%$ B 16% C 20% D 25% E 50%

1426

16. C If the areas of the original and new triangles are the same then the product of the base and the perpendicular height must be the same for the two triangles. When the base of the original triangle is increased by 25%, its value is multiplied by $\frac{5}{4}$. So if the area is to remain unchanged then the perpendicular height must be multiplied by $\frac{4}{5}$, which means that its new value is 80% of its previous value. So it is decreased by 20%.

17. How many weeks are there in $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ minutes?

A 1 B 2 C 3 D 4 E 5

17. D The number of minutes in one week is $7 \times 24 \times 60$, which may be written as $7 \times (6 \times 4) \times (5 \times 3 \times 2 \times 2) = (7 \times 6 \times 5 \times 4 \times 3 \times 2) \times 2$. So the number of weeks in $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ minutes is $8 \div 2 = 4$.

18. Consider looking from the origin (0, 0) towards all the points (m, n), where each of m and n is an integer. Some points are *hidden*, because they are directly in line with another nearer point. For example, (2, 2) is hidden by (1, 1).
How many of the points (6, 2), (6, 3), (6, 4), (6, 5) are *not* hidden points?

A 0 B 1 C 2 D 3 E 4

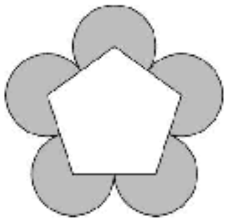
18. B The point (m, n) is hidden if and only if m and n share a common factor greater than 1. So (6, 2) is hidden by (3, 1) since 6 and 2 have common factor 2. Also (6, 3) is hidden by (2, 1) whilst (6, 4) is hidden by (3,2). However, 6 and 5 have no common factor other than 1 and therefore (6, 5) is not a hidden point.

19. Suppose that $8^m = 27$. What is the value of 4^m?

A 3 B 4 C 9 D 13.5 E there is no such m

19. C Note that $8^m = (2^3)^m = 2^{3m} = (2^m)^3$ and $27 = 3^3$; so $2^m = 3$. Therefore $4^m = 2^m \times 2^m = 9$.

20. The diagram shows a regular pentagon and five circular arcs. The sides of the pentagon have length 4. The centre of each arc is a vertex of the pentagon, and the ends of the arc are the midpoints of the two adjacent edges.



What is the total shaded area?

A 8π B 10π C 12π D 14π E 16π

20. D Each exterior angle of a regular pentagon is $\frac{1}{5} \times 360° = 72°$. So each of the five circular arcs has radius 2 and so subtends an angle of $(180 + 72)°$ at a vertex of the pentagon. Therefore the area of each of the five shaded major sectors is $\frac{252}{360} \times \pi \times 2^2 = \frac{7}{10} \times \pi \times 4 = \frac{14\pi}{5}$. So the total shaded area is 14π.

21. In King Arthur's jousting tournament, each of the several competing knights receives 17 points for every bout he enters. The winner of each bout receives an extra 3 points. At the end of the tournament, the Black Knight has exactly one more point than the Red Knight.

What is the smallest number of bouts that the Black Knight could have entered?

A 3 B 4 C 5 D 6 E 7

21. D Firstly suppose that any two knights X and Y win x and y bouts respectively and that x is at least as large as y. The difference between their total scores would be the same as if X had won $x - y$ bouts and Y had won none, since each of the separate totals would have been reduced by the same amount, namely $20y$. A similar procedure applies to losses. For example, if X won 3 and lost 6, while Y won 8 and lost 2, the difference between their total scores is the same as if X won 0 and lost 4, while Y won 5 and lost 0. In each case the difference is 32.
This argument shows that, in the case of the Black Knight, B, and the Red Knight, R, the smallest number of bouts will be achieved when one of B, R wins all his bouts and the other loses all his bouts. Also B has to score one more point than R. The possible scores for the knight who wins all his bouts are 20, 40, 60, 80, 100, 120, … . while the possible scores for the knight who loses all his bouts are 17, 34, 51, 68, 85, 102, 119, 136, … .
The first two numbers to differ by 1 are 119 and 120. Thus the Black Knight has a total of 120 corresponding to winning all of his 6 bouts and the Red Knight has a total of 119 corresponding to losing all of his 7 bouts.

22. The positive integers a, b and c are all different. None of them is a square but all the products ab, ac and bc are squares. What is the least value that $a + b + c$ can take?

A 14 B 28 C 42 D 56 E 70

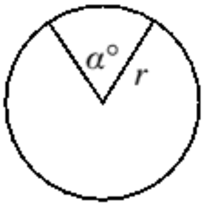
22. B Let $a = a_1a_2$ where a_2 is the largest square dividing a. Note that a_1 is then a product of distinct primes. Similarly write $b = b_1b_2$ and $c = c_1c_2$. Since ab is a square, a_1b_1 must be a square; so $a_1 = b_1 = k$ say. Similarly $c_1 = k$. The smallest possible value of k is 2 (since a is not a square); and the smallest possible values for a_2, b_2, c_2 are 1, 4 and 9 in some order. This makes $a + b + c = 2 + 8 + 18 = 28$.

23. A sector of a disc is removed by making two straight cuts from the circumference to the centre. The perimeter of the sector has the same length as the circumference of the original disc. What fraction of the area of the disc is removed?

A $\frac{\pi - 1}{\pi}$ B $\frac{1}{\pi}$ C $\frac{\pi}{360}$ D $\frac{1}{3}$ E $\frac{1}{2}$

1433

23. A Let the radius of the circle be r and let the angle of the sector be $\alpha°$.



Then the perimeter of the sector is $2r + \frac{\alpha}{360} \times 2\pi r$.
This equals $2\pi r$, the circumference of the original circle.
So $2r + \frac{\alpha}{360} \times 2\pi r = 2\pi r$.
Therefore the fraction of the area of the disc removed is $\frac{\alpha}{360} = \frac{2\pi r - 2r}{2\pi r} = \frac{\pi - 1}{\pi}$.

24. How many 4-digit integers (from 1000 to 9999) have at least one digit repeated?

A 62×72 B 52×72 C 52×82 D 42×82 E 42×92

24. A There are 9000 four-digit integers. To calculate the number of these which have four different digits, we note that we have a choice of 9 for the thousands digit. We now have a choice of 9 for the hundreds digit (since we can choose 0 as a possible digit). After these two digits have been chosen, we have a choice of 8 for the tens digit and then 7 for the units digit. So the number of four-digit numbers in which all digits are different is $9 \times 9 \times 8 \times 7$.
Therefore the number of four-digit numbers which have at least one digit repeated is $9000 - 9 \times 9 \times 8 \times 7 = 9(1000 - 9 \times 8 \times 7) =$
$9 \times 8 \times (125 - 9 \times 7) = 72 \times (125 - 63) = 72 \times 62$.

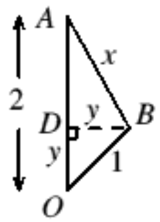
25. The diagram shows two concentric circles with radii 1 and 2 units, together with a shaded octagon, all of whose sides are equal.

What is the length of the perimeter of the octagon?

A $8\sqrt{2}$ B $8\sqrt{3}$ C $8\sqrt{3}\pi$

D $2\sqrt{5 + 2\sqrt{2}}$ E $8\sqrt{5 - 2\sqrt{2}}$

25. E Let each side of the octagon have length x. The octagon may be divided into eight triangles by joining the centre of the circle to the vertices of the octagon. One such triangle is shown. Each of these triangles has one side of length 1 (the radius of the smaller circle), one side of length 2 (the radius of the larger circle) and one side of length x. So all eight triangles are congruent. Therefore $\angle AOB = 360° \div 8 = 45°$.



Let D be the foot of the perpendicular from B to AO. Then triangle BDO is an isosceles right-angled triangle.

Let $OD = DB = y$. Applying Pythagoras' Theorem to triangle BDO:
$y^2 + y^2 = 1$. So $y = \frac{1}{\sqrt{2}}$.

Applying Pythagoras' Theorem to triangle ADB:
$x^2 = (2 - y)^2 + y^2 = \left(2 - \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 4 - 2 \times 2 \times \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2} = 5 - \frac{4}{\sqrt{2}} = 5 - 2\sqrt{2}$.

So the length of the perimeter is $8x = 8\sqrt{5 - 2\sqrt{2}}$.

(*Note that the length of AB may also be found by applying the Cosine Rule to triangle OAB.*)