Intermediate Mathematical Challenge 2012

1. How many of the following four numbers are prime?

3 33 333 3333

A 0 B 1 C 2 D 3 E 4

1211

1. B As $33 = 3 \times 11$, $333 = 3 \times 111$ and $3333 = 3 \times 1111$, none of 33, 333, 3333 is prime. So 3 is the only one of the four numbers which is prime.

2. Three positive integers are all different. Their sum is 7. What is their product?

A 12 B 10 C 9 D 8 E 5

1212

2. D The following are the only triples of positive integers which sum to 7:

$$(1, 1, 5),\ (1, 2, 4),\ (1, 3, 3);\ (2, 2, 3).$$

In only one of these are the three integers all different, so the required integers are 1, 2, 4 and their product is 8.

3. An equilateral triangle, a square and a pentagon all have the same side length. The triangle is drawn on and above the top edge of the square and the pentagon is drawn on and below the bottom edge of the square. What is the sum of the interior angles of the resulting polygon?

A $10 \times 180°$ B $9 \times 180°$ C $8 \times 180°$ D $7 \times 180°$ E $6 \times 180°$

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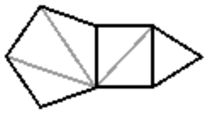
3. E The diagram shows that the interior angles of the polygon may be divided up to form the interior angles of six triangles. So their sum is $6 \times 180°$.



4. All four digits of two 2-digit numbers are different. What is the largest possible sum of two such numbers?

A 169 B 174 C 183 D 190 E 197

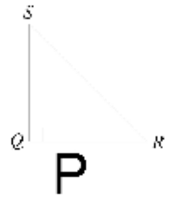
4. C The digits to be used must be 9, 8, 7, 6. If any of these were to be replaced by a smaller digit, then the sum of the two two-digit numbers would be reduced. For this sum to be as large as possible, 9 and 8 must appear in the 'tens' column rather than the 'units' column. So the largest possible sum is $97 + 86$ or $96 + 87$. In both cases the sum is 183.

5. How many minutes will elapse between 20:12 today and 21:02 tomorrow?

A 50 B 770 C 1250 D 1490 E 2450

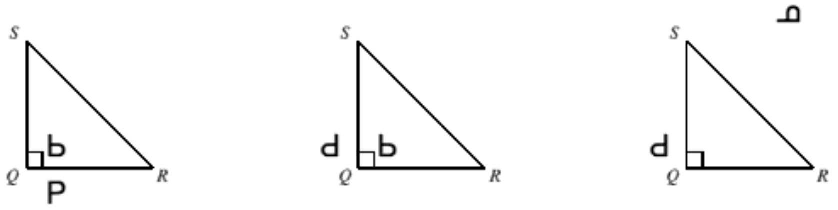
5. D The difference between the two given times is 24 hours 50 minutes. So the number of minutes that elapse is $24 \times 60 + 50 = 1440 + 50 = 1490$.

6. Triangle QRS is isosceles and right-angled.
Beatrix reflects the P-shape in the side QR to get an image.
She reflects the first image in the side QS to get a second image.
Finally, she reflects the second image in the side RS to get a third image.
What does the third image look like?



A B C D E

6. A The diagram shows the result of the successive reflections.



7. The prime numbers p and q are the smallest primes that differ by 6. What is the sum of p and q?

A 12 B 14 C 16 D 20 E 28

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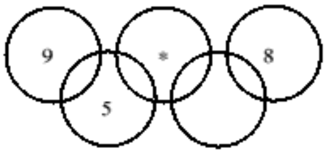
7. C The primes in question are 5 and 11. The only primes smaller than 5 are 2 and 3. However p and q cannot be 2 and 8 nor 3 and 9 since neither 8 nor 9 is prime.

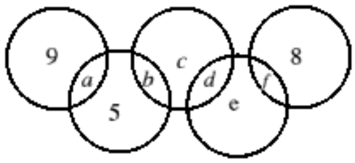
8. Seb has been challenged to place the numbers 1 to 9 inclusive in the nine regions formed by the Olympic rings so that there is exactly one number in each region and the sum of the numbers in each ring is 11. The diagram shows part of his solution.



What number goes in the region marked * ?

A 6 B 4 C 3 D 2 E 1

8. **A** Referring to the diagram, $a = 11 - 9 = 2$; $b = 11 - 5 - a = 4$; $f = 11 - 8 = 3$. So the values of c, d and e are 1, 6, 7 in some order. We need to have $b + c + d = 11$ and $d + e + f = 11$. Therefore, as $b = 4$ and $f = 3$, $c + d = 7$ and $d + e = 8$. The only solution with c, d, e chosen from 1, 6, 7 is $c = 6$, $d = 1$ and $e = 7$. So 6 must be in the region labelled *.



9. Auntie Fi's dog Itchy has a million fleas. His anti-flea shampoo claims to leave no more than 1% of the original number of fleas after use. What is the least number of fleas that will be eradicated by the treatment?

A 900 000 B 990 000 C 999 000 D 999 990 E 999 999

9. B 1% of 1 000 000 = 1 000 000 ÷ 100 = 10 000. So the least number of fleas which will be eradicated is 1 000 000 − 10 000 = 990 000.

10. An 'abundant' number is a positive integer N, such that the sum of the factors of N (excluding N itself) is greater than N. What is the smallest abundant number?

A 5 B 6 C 10 D 12 E 15

1220

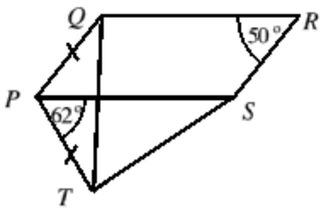
10. D The table shows the first 12 positive integers, N, and the sum, S, of the factors of N excluding N itself. As can be seen, 12 is the first value of N for which this sum exceeds N, so 12 is the smallest abundant number.

N	1	2	3	4	5	6	7	8	9	10	11	12
S	0	1	1	3	1	6	1	7	4	8	1	16

(*Note that for* $N = 6$ *the sum,* S*, also equals* 6. *For this reason,* 6 *is known as a 'perfect number'. After* 6, *the next two perfect numbers are* 28 *and* 496.)

11. In the diagram, $PQRS$ is a parallelogram; $\angle QRS = 50°$; $\angle SPT = 62°$ and $PQ = PT$.
What is the size of $\angle TQR$?

A 84° B 90° C 96° D 112° E 124°



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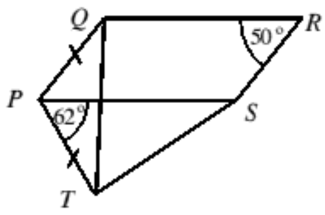
11. C Opposite angles of a parallelogram are equal, so $\angle QPS = 50°$. Therefore $\angle QPT = 112°$ and, as triangle QPT is isosceles, $\angle PQT = (180° - 112°) \div 2 = 34°$.
As $PQRS$ is a parallelogram, $\angle PQR = 180° - 50° = 130°$.
So $\angle TQR = 130° - 34° = 96°$.



12. Which one of the following has a different value from the others?

A 18% of £30 B 12% of £50 C 6% of £90 D 4% of £135 E 2% of £270

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12. B The values of the expressions are £5.40, £6.00, £5.40, £5.40, £5.40 respectively.

13. Alex Erlich and Paneth Farcas shared an opening rally of 2 hours and 12 minutes during their table tennis match at the 1936 World Games. Each player hit around 45 shots per minute. Which of the following is closest to the total number of shots played in the rally?

A 200 B 2000 C 8000 D 12 000 E 20 000

13. D In the rally, approximately 90 shots were hit per minute for a total of 132 minutes. As $90 \times 130 = 11\,700$, D is the best alternative.

14. What value of x makes the mean of the first three numbers in this list equal to the mean of the last four?

15 5 x 7 9 17

A 19 B 21 C 24 D 25 E 27

14. A The mean of the first three numbers is $\frac{1}{3}(20 + x)$; the mean of the last four numbers is $\frac{1}{4}(33 + x)$. Therefore $4(20 + x) = 3(33 + x)$, that is $80 + 4x = 99 + 3x$, so $x = 99 - 80 = 19$.

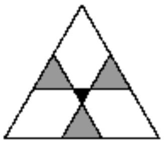
15. Which of the following has a value that is closest to 0?

A $\frac{1}{2}+\frac{1}{3}\times\frac{1}{4}$ B $\frac{1}{2}+\frac{1}{3}\div\frac{1}{4}$ C $\frac{1}{2}\times\frac{1}{3}\div\frac{1}{4}$ D $\frac{1}{2}-\frac{1}{3}\div\frac{1}{4}$ E $\frac{1}{2}-\frac{1}{3}\times\frac{1}{4}$

15. E $\frac{1}{2}+\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}+\frac{1}{12}=\frac{7}{12}$; $\frac{1}{2}+\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}+\frac{1}{3}\times\frac{4}{1}=\frac{1}{2}+\frac{4}{3}=\frac{11}{6}$; $\frac{1}{2}\times\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}\times\frac{1}{3}\times\frac{4}{1}=\frac{2}{3}$; $\frac{1}{2}-\frac{1}{3}\div\frac{1}{4}=\frac{1}{2}-\frac{1}{3}\times\frac{4}{1}=\frac{1}{2}-\frac{4}{3}=-\frac{5}{6}$; $\frac{1}{2}-\frac{1}{3}\times\frac{1}{4}=\frac{1}{2}-\frac{1}{12}=\frac{5}{12}$.

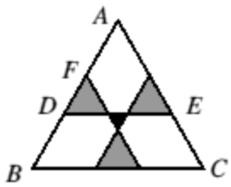
Of the fractions $\frac{7}{12}, \frac{11}{6}, \frac{2}{3}, -\frac{5}{6}, \frac{5}{12}$, the closest to 0 is $\frac{5}{12}$.

16. The diagram shows a large equilateral triangle divided by three straight lines into seven regions. The three grey regions are equilateral triangles with sides of length 5 cm and the central black region is an equilateral triangle with sides of length 2 cm.
What is the side length of the original large triangle?



A 18 cm B 19 cm C 20 cm D 21 cm E 22 cm

16. B As triangle ABC is equilateral, $\angle BAC = 60°$. Since the grey triangles are equilateral, $\angle ADE = 60°$, so the triangle ADE is equilateral. The length of the side of this triangle is equal to the length of $DE = (5 + 2 + 5)$ cm $= 12$ cm. So $AF = AD - FD = (12 - 5)$ cm $= 7$ cm. By a similar argument, we deduce that $BD = 7$ cm, so the length of the side of triangle $ABC = (7 + 5 + 7)$ cm $= 19$ cm.



17. The first term of a sequence of positive integers is 6. The other terms in the sequence follow these rules:

if a term is even then divide it by 2 to obtain the next term;

if a term is odd then multiply it by 5 and subtract 1 to obtain the next term.

For which values of n is the nth term equal to n?

A 10 only B 13 only C 16 only D 10 and 13 only E 13 and 16 only

17. E The terms of the sequence are 6, 3, 14, 7, 34, 17, 84, 42, 21, 104, 52, 26, 13, 64, 32, 16, 8, 4, 2, 1, 4, 2, 1, … . As can be seen, there will now be no other terms in the sequence other than 4, 2 and 1. It can also be seen that the only values of n for which the n th term = n are 13 and 16.

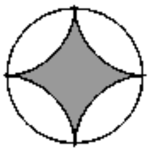
18. Peri the winkle starts at the origin and slithers anticlockwise around a semicircle with centre (4, 0). Peri then slides anticlockwise around a second semicircle with centre (6, 0), and finally clockwise around a third semicircle with centre (3, 0).
Where does Peri end this expedition?

A (0, 0) B (1, 0) C (2, 0) D (4, 0) E (6, 0)

1228

18. C After traversing the first semicircle, Peri will be at the point (8, 0); after the second semicircle Peri will be at (4, 0) and after the third semicircle, Peri will be at the point (2, 0).

19. The shaded region shown in the diagram is bounded by four arcs, each of the same radius as that of the surrounding circle. What fraction of the surrounding circle is shaded?



A $\frac{4}{\pi} - 1$ B $1 - \frac{\pi}{4}$ C $\frac{1}{2}$ D $\frac{1}{3}$ E it depends on the radius of the circle

19. A The diagram shows the original diagram enclosed within a square of side $2r$, where r is the radius of the original circle. The unshaded area of the square consists of four quadrants (quarter circles) of radius r. So the shaded area is $4r^2 - \pi r^2 = r^2(4 - \pi)$. Therefore the required fraction is

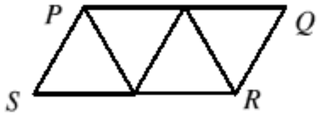
$$\frac{r^2(4 - \pi)}{\pi r^2} = \frac{4 - \pi}{\pi} = \frac{4}{\pi} - 1.$$

20. A rectangle with area 125 cm^2 has sides in the ratio 4:5. What is the perimeter of the rectangle?

A 18 cm B 22.5 cm C 36 cm D 45 cm E 54 cm

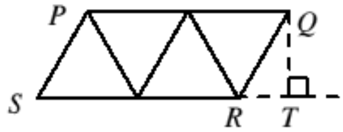
20. D Let the sides of the rectangle, in cm, be $4x$ and $5x$.
Then the area of the square is $4x \times 5x$ cm^2 = $20x^2$ cm^2. So $20x^2 = 125$, that is $x^2 = \frac{25}{4}$.
Therefore $x = \pm\frac{5}{2}$, but x cannot be negative so $x = \frac{5}{2}$ and so the sides of the rectangle are 10 cm and 12.5 cm. Hence the rectangle has perimeter 45 cm.

21. The parallelogram $PQRS$ is formed by joining together four equilateral triangles of side 1 unit, as shown.
What is the length of the diagonal SQ?



A $\sqrt{7}$ B $\sqrt{8}$ C 3 D $\sqrt{6}$ E $\sqrt{5}$

21. A In the diagram, T is the foot of the perpendicular from Q to SR produced. Angles PQR and QRT are alternate angles between parallel lines so $\angle QRT = 60°$. Triangle QRT has interior angles of $90°$, $60°$, $30°$ so it may be thought of as being half of an equilateral triangle of side 1 unit, since the length of QR is 1 unit. So the lengths of RT and QT are $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ units respectively.



Applying Pythagoras' Theorem to ΔQST, $SQ^2 = ST^2 + QT^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4} + \frac{3}{4} = 7$.
So the length of SQ is $\sqrt{7}$ units.

22. What is the maximum possible value of the median number of cups of coffee bought per customer on a day when Sundollars Coffee Shop sells 477 cups of coffee to 190 customers, and every customer buys at least one cup of coffee?

A 1.5 B 2 C 2.5 D 3 E 3.5

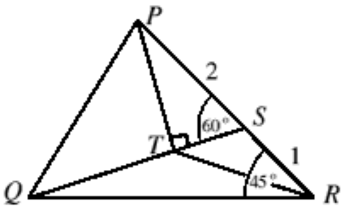
22. E The options given imply that the median cannot exceed 3.5, so we first look to see if the median can have this value. The median of 190 numbers is $\frac{1}{2}(a + b)$ where, when the numbers are arranged in increasing order, a is the 95th number and b is the 96th number in the list. In this problem, a and b are positive integers with $a \leqslant b$, so if $\frac{1}{2}(a + b) = 3.5$ then $a + b = 7$. The smallest total of all 190 numbers with median 3.5 occurs when $a = 3, b = 4$ and the numbers, in increasing order, are

$$\underbrace{1,1,1,\ \dots\ ,1}_{94},\ 3,\ \underbrace{4,4,4,\ \dots\ ,4}_{95}.$$

In this case their total is $94 \times 1 + 3 + 95 \times 4 = 94 + 3 + 380 = 477$, which is the total number of cups of coffee given in the question. So 3.5 is a possible value for the median. Note also, that, if the median were greater than 3.5, then at least one of the numbers in the above list would need to be larger and so the total would be larger than 477. So 3.5 really is the maximum possible value of the median.

23. In triangle PQR, $PS = 2$; $SR = 1$; $\angle PRQ = 45°$; T is the foot of the perpendicular from P to QS and $\angle PST = 60°$. What is the size of $\angle QPR$?

A 45° B 60° C 75° D 90° E 105°



1233 ©UKMT

23. C As in the solution for Q21, ΔPTS may be thought of as half an equilateral triangle, so TS has length 1 unit. Therefore ΔSRT is isosceles and, as $\angle TSR = 120°$, $\angle SRT = \angle STR = 30°$. So $\angle TRQ = 45° - 30° = 15°$. Using the exterior angle theorem in ΔTQR, $\angle TQR = \angle STR - \angle TRQ = 30° - 15° = 15°$. So ΔTQR is isosceles with $TQ = TR$. However, ΔPRT is also isosceles with $PT = TR$ since $\angle PRT = \angle TPR = 30°$. Therefore $TQ = TP$, from which we deduce that PQT is an isosceles right-angled triangle in which $\angle PQT = \angle QPT = 45°$. So $\angle QPR = \angle QPT + \angle TPS = 45° + 30° = 75°$.

24. All the positive integers are written in the cells of a square grid. Starting from 1, the numbers spiral anticlockwise. The first part of the spiral is shown in the diagram.

What number will be immediately below 2012?

A 1837 B 2011 C 2013 D 2195 E 2210

					…	32	31	
		17	16	15	14	13	30	
		18	5	4	3	12	29	
		19	6	1	2	11	28	
		20	7	8	9	10	27	
		21	22	23	24	25	26	

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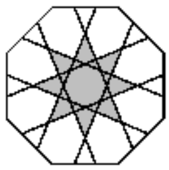
24. D The nature of the spiral means that 4 is in the top left-hand corner of a 2×2 square of cells, 9 is in the bottom right-hand corner of a 3×3 square of cells, 16 is in the top left-hand corner of a 4×4 square of cells and so on. To find the position of 2012 in the grid, we note that $45^2 = 2025$ so 2025 is in the bottom right-hand corner of a 45×45 square of cells and note also that $47^2 = 2209$. The table below shows the part of the grid in which 2012 lies. The top row shows the last 15 cells in the bottom row of a 45×45 square of cells, whilst below it are the last 16 cells in the bottom row of a 47×47 square of cells.

2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025	
2194	2195	2196	2197	2198	2199	2200	2201	2202	2203	2204	2205	2206	2207	2208	2209

So 2195 lies immediately below 2012.

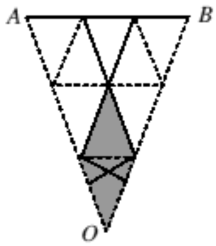
25. The diagram shows a ceramic design by the Catalan architect Antoni Gaudi. It is formed by drawing eight lines connecting points which divide the edges of the outer regular octagon into three equal parts, as shown.

What fraction of the octagon is shaded?



A $\frac{1}{5}$ B $\frac{2}{9}$ C $\frac{1}{4}$ D $\frac{3}{10}$ E $\frac{5}{16}$

25. B The diagram shows part of the ceramic.



A and *B* are vertices of the outer octagon, which has *O* at its centre. The lines *OA*, *OB*, two lines which are parallel to *AB* and lines parallel to *OA* and *OB* respectively have been added. As can be seen, these lines divide ΔOAB into nine congruent triangles. The shaded portion of triangle has area equal to that of two of the triangles. So $\frac{2}{9}$ of the area of ΔOAB has been shaded. Now the area of the outer octagon is eight times the area of ΔOAB and the area of shaded portion of the design is eight times the area of the shaded portion of ΔOAB so the fraction of the octagon which is shaded is also $\frac{2}{9}$.