Intermediate Mathematical Challenge 2011

1. What is the value of $4.5 \times 5.5 + 4.5 \times 4.5$?

A 36.5 B 45 C 50 D 90 E 100

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1. B $4.5 \times 5.5 + 4.5 \times 4.5 = 4.5(5.5 + 4.5) = 4.5 \times 10 = 45.$

2. To find the diameter in mm of a Japanese knitting needle, you multiply the size by 0.3 and add 2.1. What is the diameter in mm of a size 5 Japanese knitting needle?

A 3.6 B 7.4 C 10.8 D 12 E 17.1

2. A The diameter is $(5 \times 0.3 + 2.1)$ mm $= (1.5 + 2.1)$ mm $= 3.6$ mm.

3. The consecutive digits 1, 2, 3, 4 in that order can be arranged to make the correct division, $12 \div 3 = 4$. One *other* sequence of four consecutive digits p, q, r, s makes a correct division, 'pq' $\div r = s$. What is the value of s in this case?

A 4 B 5 C 6 D 7 E 8

3. **E** $12 \div 3 = 4$, but this is the example given in the question. $23 \div 4 \neq 5$; $34 \div 5 \neq 6$; $45 \div 6 \neq 7$. However, $56 \div 7 = 8$, so $s = 8$.
{*Note also that* $67 \div 8 \neq 9$.}

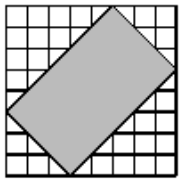

4. The angles of a triangle are in the ratio 2:3:5. What is the difference between the largest angle and the smallest angle?

A 9° B 18° C 36° D 45° E 54°

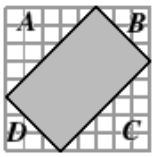
4. **E** The difference between the angles is $\left(\frac{5}{10} - \frac{2}{10}\right) \times 180°$.

5. The diagram shows a rectangle placed on a grid of 1 cm ×1 cm squares.
What is the area of the rectangle in cm^2?

A 15 B $22\frac{1}{2}$ C 30 D 36 E 45



5. C Triangles A and C are each of area $\frac{1}{2} \times 5 \times 5$ cm^2.
Triangles B and D are each of area $\frac{1}{2} \times 3 \times 3$ cm^2.
So the shaded area is $[64 - (25 + 9)]$ $cm^2 = 30$ cm^2.



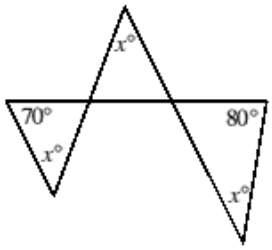
6. When I glanced at my car milometer it showed 24942, a palindromic number. Two days later, I noticed that it showed the next palindromic number. How many miles did my car travel in those two days?

A 100 B 110 C 200 D 220 E 1010

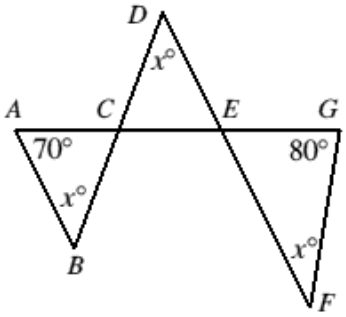
6. **B** The next palindromic number after 24942 is 25052, so the car travelled 110 miles in the two days.

7. What is the value of x in this diagram?

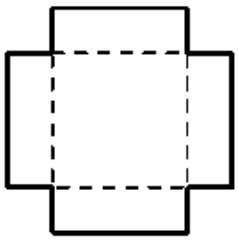
A 30 B 35 C 40 D 45 E 50



7. **A** Alternate angles BDF and DFG are equal, so lines BD and FG are parallel. Therefore $\angle BCA = \angle FGC = 80°$ (corresponding angles). Consider triangle ABC: $x + 70 + 80 = 180$, so $x = 30$.



8. A square piece of card has a square of side 2 cm cut out from each of its corners. The remaining card is then folded along the dotted lines shown to form an open box whose total internal surface area is 180 cm^2.



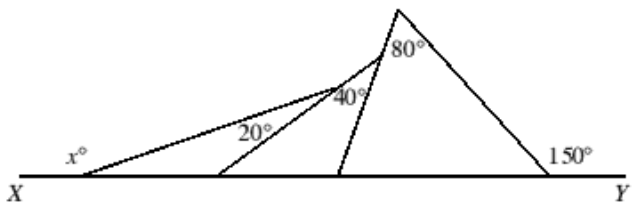
What is the volume of the open box in cm^3?

A 100 B 128 C 162 D 180 E 200

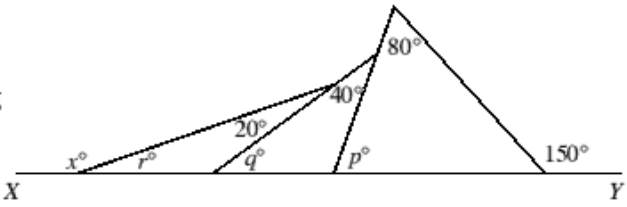
8. E The base of the open box is a square. Let its side be of length x cm. Then the total surface area of the box in cm^2 is $x^2 + 4 \times 2x = x^2 + 8x$. Hence $x^2 + 8x = 180$, that is $x^2 + 8x - 180 = 0$. Therefore $(x+18)(x-10) = 0$, which gives $x = -18$ or $x = 10$. As x is positive, it may be deduced that the open box has dimensions 10 cm × 10 cm × 2 cm. So its volume is 200 cm^3.

9. In the diagram, XY is a straight line. What is the value of x?

A 170 B 160 C 150 D 140 E 130



9. A In a triangle, an exterior angle is equal to the sum of the two interior, opposite angles. Repeatedly applying this theorem:
$p = 150 - 80 = 70$;
$q = p - 40 = 30$; $r = q - 20 = 10$.
Therefore $x = 180 - r = 170$.



10. Merlin magically transforms a 6 tonne monster into mice with the same total mass. Each mouse has a mass of 20g. How many mice does Merlin make?

A 30 B 300 C 3000 D 30 000 E 300 000

10. E One tonne = 1000kg = 1 000 000g. So the number of mice is $6\,000\,000 \div 20 = 300\,000$.

11. What is the value of $19\frac{1}{2} \times 20\frac{1}{2}$?

A 250 B $380\frac{1}{4}$ C $390\frac{1}{4}$ D 395 E $399\frac{3}{4}$

11. E $19\frac{1}{2} \times 20\frac{1}{2} = \left(20 - \frac{1}{2}\right) \times \left(20 + \frac{1}{2}\right) = 20^2 - \left(\frac{1}{2}\right)^2 = 400 - \frac{1}{4} = 399\frac{3}{4}$.

12. What is the sum of the first 2011 digits when $20 \div 11$ is written as a decimal?

A 6013 B 7024 C 8035 D 9046 E 10057

12. D $20 \div 11 = 1\frac{9}{11} = 1.818181\ldots$. So the first 2011 digits are 1006 '1's and 1005 '8's. Therefore the required total is $1006 \times 1 + 1005 \times 8 = 1006 + 8040 = 9046$.

13. The three blind mice stole a piece of cheese. In the night, the first mouse ate $\frac{1}{3}$ of the cheese. Later, the second mouse ate $\frac{1}{3}$ of the remaining cheese. Finally, the third mouse ate $\frac{1}{3}$ of what was then left of the cheese.
Between them, what fraction of the cheese did they eat?

A $\frac{16}{27}$ B $\frac{17}{27}$ C $\frac{2}{3}$ D $\frac{19}{27}$ E $\frac{20}{27}$

13. D After the first mouse has eaten, $\frac{2}{3}$ of the cheese remains. After the second mouse has eaten, $\frac{2}{3}$ of $\frac{2}{3}$, that is $\frac{4}{9}$, of the cheese remains. Finally, after the third mouse has eaten, $\frac{2}{3}$ of $\frac{4}{9}$, that is $\frac{8}{27}$, of the cheese remains. So the mice ate $\frac{19}{27}$ of the cheese.

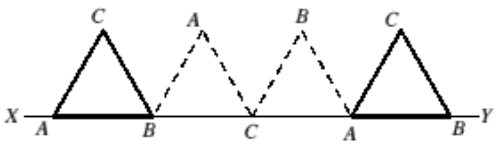
14. The number 6 lies exactly halfway between 3 and 3^2. Which of the following is not halfway between a positive integer and its square?

A 3 B 10 C 15 D 21 E 30

14. E $3 = \frac{2+2^2}{2}$; $10 = \frac{4+4^2}{2}$; $15 = \frac{5+5^2}{2}$; $21 = \frac{6+6^2}{2}$. However $\frac{7+7^2}{2} = 28$ and $\frac{8+8^2}{2} = 36$, so 30 is not exactly halfway between a positive integer and its square.

(*Note that every number which is exactly halfway between a positive integer and its square is a triangle number. Can you explain why this is so?*)

15. The equilateral triangle ABC has sides of length 1 and AB lies on the line XY. The triangle is rotated clockwise around B until BC lies on the line XY. It is then rotated similarly around C and then about A as shown in the diagram.



What is the length of the path traced out by point C during this sequence of rotations?

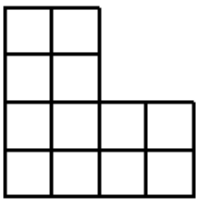
A $\frac{4\pi}{3}$ B $2\sqrt{3}$ C $\frac{8\pi}{3}$ D 3 E $\frac{2\pi}{3}$

15. A In each rotation which C makes, the radius of the arc it describes is 1 unit. In the first rotation, C turns through an angle of 120°, so it moves a distance $\frac{1}{3} \times 2 \times \pi \times 1$, that is $\frac{2\pi}{3}$. As it is the centre of the second rotation, C does not move during it. In the third rotation, C again turns through an angle of 120°, so the total distance travelled is $2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$.

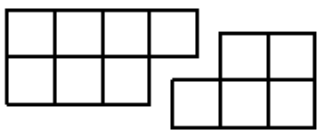
16. The diagram shows an L-shape divided into 1×1 squares. Gwyn cuts the shape along some of the lines shown to make two pieces, neither of which is a square. She then uses the pieces to form a 2×6 rectangle.

What is the difference between the areas of the two pieces?

A 0 B 1 C 2 D 3 E 4



16. C The L-shape needs to be divided as shown since neither of the pieces is to be a square. Notice that one of the pieces must be turned over. The difference between the areas of the two pieces is $7 - 5 = 2$.

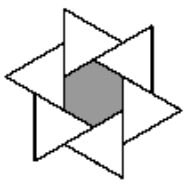


17. A shop advertised "Everything half price in our sale", but also now advertises that there is "An additional 15% off sale prices". Overall, this is equivalent to what reduction on the original prices?

A 7.5% B 35% C 57.5% D 65% E 80%

17. C The reduction of 15% off sale prices is equal to a reduction of 7.5% off the original prices. Therefore the total reduction on the original prices is $(50 + 7.5)\% = 57.5\%$.

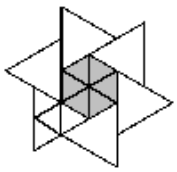
18. The diagram contains six equilateral triangles with sides of length 2 and a regular hexagon with sides of length 1.



What fraction of the whole shape is shaded?

A $\frac{1}{8}$ B $\frac{1}{7}$ C $\frac{1}{6}$ D $\frac{1}{5}$ E $\frac{1}{4}$

18. D As the diagram shows, each equilateral triangle may be divided into four equilateral triangles of side 1, whilst the hexagon may be divided into six equilateral triangles of side 1.

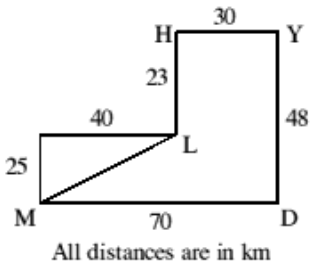


Therefore the fraction of the whole shape which is shaded is $\frac{6}{6 \times 4 + 6} = \frac{6}{30} = \frac{1}{5}$.

19. Harrogate is 23km due north of Leeds, York is 30km due east of Harrogate, Doncaster is 48km due south of York, and Manchester is 70km due west of Doncaster. To the nearest kilometre, how far is it from Leeds to Manchester, as the crow flies?

A 38km B 47km C 56km D 65km E 74km

19. B As can be seen from the diagram, Manchester is 40km west of Leeds and 25 km south of it.



All distances are in km

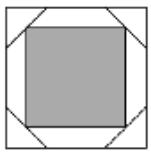
Therefore, by Pythagoras' Theorem, the distance in km from Leeds to Manchester as the crow flies is $\sqrt{25^2 + 40^2} = 5\sqrt{5^2 + 8^2} = 5\sqrt{89}$.
Now $\sqrt{81} = 9$ and $\sqrt{100} = 10$, so $9 < \sqrt{89} < 10$.
This means that the required distance is between 45km and 50km and, of the options given, only 47km (corresponding to the approximation $\sqrt{89} \approx 9.4$) lies in this interval.
(*Please note that the distances given in this problem are all approximate.*)

20. Max and his dog Molly set out for a walk. Max walked up the road and then back down again, completing a six mile round trip. Molly, being an old dog, walked at half Max's speed. When Max reached the end of the road, he turned around and walked back to the starting point, at his original speed. Part way back he met Molly, who then turned around and followed Max home, still maintaining her original speed. How far did Molly walk?

A 1 mile B 2 miles C 3 miles D 4 miles E 5 miles

20. D Let Max and Molly meet after the latter has travelled x miles. Then Max has travelled $(6 - x)$ miles. So $6 - x = 2x$, thus $x = 2$. Therefore Molly walks a total of 4 miles.

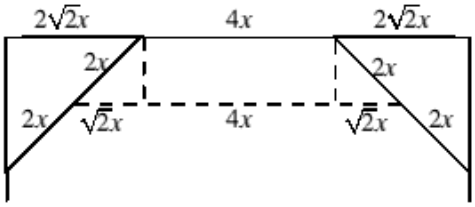
21. A regular octagon is placed inside a square, as shown. The shaded square connects the midpoints of four sides of the octagon.



What fraction of the outer square is shaded?

A $\sqrt{2}-1$ B $\frac{1}{2}$ C $\frac{\sqrt{2}+1}{4}$ D $\frac{\sqrt{2}+2}{5}$ E $\frac{3}{4}$

21. B Let $4x$ be the length of each side of the regular octagon. The diagram shows part of the figure. The four triangles shown in the diagram are all isosceles right-angled triangles. In such triangles the ratio of the length of the hypotenuse to the length of the shorter sides is $\sqrt{2} : 1$.



So in the larger triangles which have hypotenuse of length $4x$, the length of the shorter sides is $2\sqrt{2}x$, whilst the smaller triangles with hypotenuse $2x$ have shorter sides of length $\sqrt{2}x$.

Therefore the shaded square in the question has side of length $(4 + 2\sqrt{2})x$.

The length of the side of the outer square is $(4\sqrt{2} + 4)x = \sqrt{2}(4 + 2\sqrt{2})x$.

Therefore the two squares have sides in the ratio $1 : \sqrt{2}$, which means that their areas have ratio $1 : 2$.

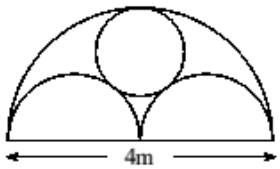
22. You are given that $5^p = 9$, $9^q = 12$, $12^r = 16$, $16^s = 20$ and $20^t = 25$. What is the value of $pqrst$?

A 1 B 2 C 3 D 4 E 5

22. B $5^p = 9$ and $9^q = 12$. Therefore $(5^p)^q = 12$, that is $5^{pq} = 12$. Similarly, as $12^r = 16$ then $5^{pqr} = 16$, as $16^s = 20$ then $5^{pqrs} = 20$ and, finally, as $20^t = 25$ then $5^{pqrst} = 25$. Therefore $pqrst = 2$.

23. A window frame in Salt's Mill consists of two equal semicircles and a circle inside a large semicircle with each touching the other three as shown. The width of the frame is 4m.

What is the radius of the circle, in metres?



A $\frac{2}{3}$ B $\frac{\sqrt{2}}{2}$ C $\frac{3}{4}$ D $2\sqrt{2}-1$ E 1

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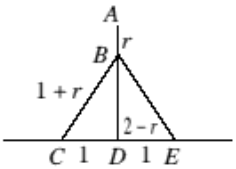
23. A Points A, B, C, D and E are, respectively: the point where the large semicircle and the circle touch, the centre of the circle, the centre of the left-hand semicircle, the centre of the large semicircle and the centre of the right-hand semicircle. The radius of the circle is r m. In triangle BCD, CD has length 1m, BD has length $(2-r)$ m, since AD is a radius of the semicircle of diameter 4m, and BC has length $(1+r)$ m, since it is the sum of the radii of the left-hand semicircle and the circle. Therefore, by Pythagoras' Theorem: $(1+r)^2 = 1^2 + (2-r)^2$, that is $1+2r+r^2 = 1+4-4r+r^2$. So $6r = 4$ and the radius of the circle is $\frac{2}{3}$m.



(*Note that triangle BCD is a* 3,4,5 *triangle with sides* $\frac{3}{5}$m, $\frac{4}{5}$m *and* $\frac{5}{5}$m.)

24. Given any positive integer n, Paul adds together the distinct factors of n, other than n itself. Which of these numbers can never be Paul's answer?

A 1 B 3 C 5 D 7 E 9

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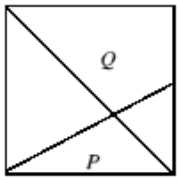
24. C Firstly we give examples to show that Paul's answer could have been any of A, B, D or E.

A: If n is prime then the only factor of n other than itself is 1.
B: Take $n = 4$. Its factors are 1, 2 and 4, and $1 + 2 = 3$.
D: Take $n = 8$. Its factors are 1, 2, 4 and 8, and $1 + 2 + 4 = 7$.
E: Take $n = 15$. Its factors are 1, 3, 5 and 15, and $1 + 3 + 5 = 9$.

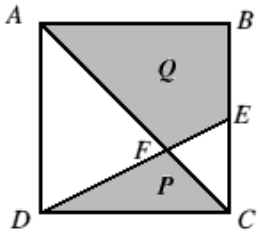
We now show that Paul's answer cannot be C. If the sum were 5, then the factors of n other than itself would have to be 1 and 4, as we are not allowed to repeat any number in the sum. However, if 4 is a factor of n, then 2 is also a factor, which produces a contradiction.

25. The diagram shows a square, a diagonal and a line joining a vertex to the midpoint of a side. What is the ratio of area P to area Q?



A $1:\sqrt{2}$ B $2:3$ C $1:2$ D $2:5$ E $1:3$

25. D Let triangle CEF have area a. Note that $\angle AFD = \angle CFE$ (vertically opposite angles) and $\angle DAF = \angle ECF$ (alternate angles), so triangles ADF and CEF are similar.



Note also that the side AD is twice the length of the corresponding side CE.

Hence:

(i) triangle ADF has area $4a$ and;

(ii) AF has twice the length of the corresponding side CF.

View AF and CF as bases of the triangles ADF and CDF (which then share the same height). Therefore, by (ii), triangle ADF has twice the area of triangle CDF (area $\boldsymbol{P}$), which thus is $2a$.

The area of triangle ACD is $6a$; so that of triangle ABC is also $6a$ and that of area $\boldsymbol{Q}$ is $5a$. So the required ratio is 2 : 5.