Intermediate Mathematical Challenge 2008

1. How many hours are there in this week?

A 24 B 70 C 84 D 148 E 168

1. E The clocks do not go forward or back this week, so there are seven 24-hour days, that is 168 hours.

2. Which is the largest prime number that divides exactly into the number equal to $2 + 3 + 5 \times 7$?

A 2 B 3 C 5 D 7 E 11

2. C $2 + 3 + 5 \times 7 = 5 + 35 = 40$. As $40 = 2^3 \times 5$, the largest prime number which divides exactly into it is 5.

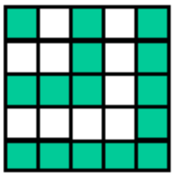
3. What is the value of $0.75 \div \frac{3}{4}$?

A 0.5 B 1 C 1.5 D 2 E 2.5

3. B $0{\cdot}75 \div \frac{3}{4} = \frac{3}{4} \div \frac{3}{4} = 1.$

4. What percentage of the large 5×5 square is shaded?

A 40% B 60% C $66\frac{2}{3}$% D 75% E 80%



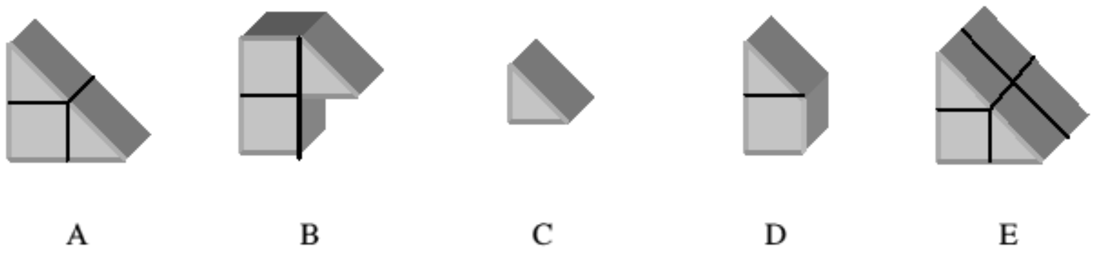
4. B The large square is made up of 25 small squares, 15 of which are shaded. So $\frac{15}{25}$ of the large square is shaded, corresponding to $\frac{15}{25} \times 100\% = 60\%$.

5. Which of the following is *not* equal to a whole number?

A $\frac{594}{5+9+4}$ B $\frac{684}{6+8+4}$ C $\frac{756}{7+5+6}$ D $\frac{873}{8+7+3}$ E $\frac{972}{9+7+2}$

5. D In each of the fractions, the denominator is 18 and the sum of the digits of the numerator is also 18. So every numerator is a multiple of 9 and the even numerators are also multiples of 18. However, 873 is not a multiple of 18, so $\frac{873}{8+7+3}$ is the only expression not equal to a whole number.

6. Four of these shapes can be placed together to make a cube. Which is the odd one out?



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6. B Let shape C have width 1 unit, height 1 unit and depth 1 unit. Then the volumes, in units^3, of the five shapes are: A 2, B $2\frac{1}{2}$, C $\frac{1}{2}$, D $1\frac{1}{2}$, E 4. These total $10\frac{1}{2}$ units^3, so we may deduce that the cube formed by the four shapes will have side 2 units and volume 8 units^3. Hence B is the shape which is not required. The cube may be formed by placing C next to D to form a shape identical to A. This combination is then placed alongside A to form a shape identical to E. If shape E is now rotated through 180° about a suitable axis, it may be placed with the combination of shapes A, C and D to form a cube.

7. The square of a non-zero number is equal to 70% of the original number. What is the original number?

A 700 B 70 C 7 D 0.7 E 0.07

7. D Let the original number be x. Then $x^2 = 0{\cdot}7x$, that is $x(x - 0{\cdot}7) = 0$. So $x = 0$ or $x = 0{\cdot}7$, but as x is non-zero it is $0{\cdot}7$.

8. In a certain year, there were exactly four Tuesdays and exactly four Fridays in October. On what day of the week did Halloween, October 31st, fall that year?

A Monday B Wednesday C Thursday D Saturday E Sunday

0818

8. A October has 31 days so, in any year, in October there are three days of the week which occur five times and four days which occur four times. As there were four Tuesdays and four Fridays, there could not have been five Wednesdays or five Thursdays, so the days which occurred five times were Saturday, Sunday and Monday. Hence October 1st fell on a Saturday, which means that October 31st was a Monday.

9. A solid wooden cube is painted blue on the outside. The cube is then cut into 27 smaller cubes of equal size. What fraction of the total surface area of these new cubes is blue?

A $\frac{1}{6}$ B $\frac{1}{5}$ C $\frac{1}{4}$ D $\frac{1}{3}$ E $\frac{1}{2}$

0819

9. D Let the smaller cubes have side of length 1 unit. So the original cube had side of length 3 units and hence a surface area of 54 units^2, all of which was painted blue. The total surface area of the 27 small cubes is 27×6 units^2, that is 162 units^2. So the required fraction is $\frac{54}{162} = \frac{1}{3}$.

10. Two sides of a triangle have lengths 6 cm and 5 cm. Perry suggests the following possible values for the perimeter of the triangle: (i) 11 cm (ii) 15 cm (iii) 24 cm.
Which of Perry's suggestions could be correct?

A (i) only B (i) or (ii) C (ii) only D (ii) or (iii) E (iii) only

10. C In every triangle the length of the longest side is less than the sum of the lengths of the two other sides. So if the triangle has sides of length 5 cm and 6 cm, then the length of the third side is greater than 1 cm, but less than 11 cm. Hence the perimeter, p cm, of the triangle satisfies $12 < p < 22$. So 15 cm is the only one of Perry's suggested values which could be correct.

11. | S is 25% of 60 | 60 is 80% of U | 80 is M% of 25 |

What is $S + U + M$?

A 100 B 103 C 165 D 330 E 410

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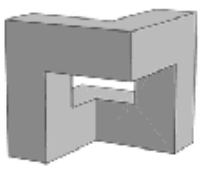
11. E $S = 25\%$ of $60 = 15$. $U = \frac{60}{0{\cdot}8} = 75$. $M = \frac{80}{0{\cdot}25} = 320$. So $S + U + M = 410$.

12. The sculpture 'Cubo Vazado' [Emptied Cube] by the Brazilian artist Franz Weissmann is formed by removing cubical blocks from a solid cube to leave the symmetrical shape shown. If all the edges have length 1, 2 or 3 units, what is the surface area of the sculpture in square units?



A 36 B 42 C 48 D 54 E 60

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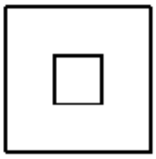
12. C In each of 6 possible directions, the view of the sculpture is as shown, with the outer square having side 3 units and the inner square having side 1 unit.
So the surface area of the sculpture is $6 \times 8\,\text{units}^2 = 48\,\text{units}^2$.



13. The mean of a sequence of 64 numbers is 64. The mean of the first 36 numbers is 36. What is the mean of the last 28 numbers?

A 28 B 44 C 72 D 100 E 108

13. D The sum of all 64 numbers is $64 \times 64 = 64^2$. The sum of the first 36 numbers is $36 \times 36 = 36^2$. So the sum of the remaining 28 numbers is $64^2 - 36^2 = (64 + 36)(64 - 36) = 2800$. Therefore the mean of these 28 numbers is 100.

14. Sam is holding two lengths of rope by their mid-points. Pat chooses two of the loose ends at random and ties them together.
What is the probability that Sam now holds one untied length of rope and one tied loop of rope?

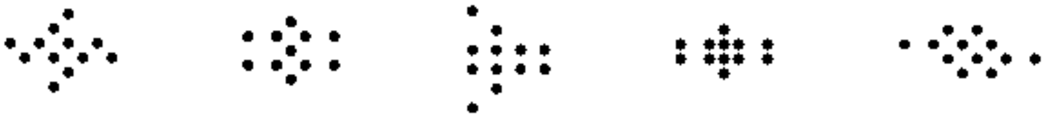
A $\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{4}$ D $\frac{1}{5}$ E $\frac{1}{6}$

0824

14. B Let A and B denote the ends of the first length of rope and C and D denote the ends of the second length of rope. Then Pat chooses one of 6 different possible combinations: (A, B), (A, C), (A, D), (B, C), (B, D), (C, D). Sam now holds one untied length of rope and one tied loop of rope if, and only if, Pat has chosen (A, B) or (C, D) so the required probability is $\frac{2}{6} = \frac{1}{3}$.

(*Alternatively: Irrespective of whichever end Pat chooses first, Sam will hold one untied length of rope and one tied loop of rope if, and only if, Pat now chooses a particular one of the three remaining ends, namely the other end of the same rope. As each of the three ends is equally likely to be chosen, the required probability is* $\frac{1}{3}$.)

15. A designer wishes to use two copies of the logo shown on the right to create a pattern, without any of the dots overlapping. Which one of the following could be made?



A B C D E

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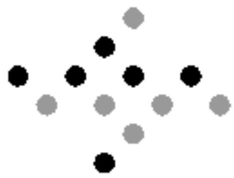
15. A Notice that a single copy of the logo consists of four dots which lie in a straight line plus two other dots which lie on the perpendicular bisector of this line. These two dots are not evenly spaced above and below the line of four dots. Of the options given, only A has two lines of four dots with two more dots in the correct positions relative to each line.



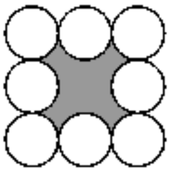
16. The first two terms of a sequence are $\frac{2}{3}$ and $\frac{4}{5}$. Each term after the second term is the average (mean) of the two previous terms. What is the fifth term in the sequence?

A $\frac{5}{34}$ B $\frac{1}{2}$ C $\frac{10}{13}$ D $\frac{3}{4}$ E $\frac{10}{11}$

0826

16. D The problem may be solved by firstly calculating the third and fourth terms of the sequence, but an algebraic method does reduce the amount of calculation involved. Let the first two terms be x and y respectively. Then the third term is $\frac{1}{2}(x + y)$, whilst the fourth term is $\frac{1}{4}(x + 3y)$. So the fifth term is $\frac{1}{8}(3x + 5y)$. Putting $x = \frac{2}{3}$ and $y = \frac{4}{5}$, we obtain $\frac{2 + 4}{8} = \frac{3}{4}$.

17. The shaded region is bounded by eight equal circles with centres at the corners and midpoints of the sides of a square. The perimeter of the square has length 8. What is the length of the perimeter of the shaded region?



A π B 2π C 8 D 3π E 4π

17. D As the perimeter of the square has length 8, the square has side length 2. So the diameter of each of the circles is 1. The perimeter of the shaded region consists of four semi-circular arcs and four quarter-circle arcs, so it has length equal to three times the circumference of one circle, that is 3π.

18. In the calculation $1003 \div 4995 = 0.2\dot{0}0\dot{8}$, the number $0.2\dot{0}0\dot{8}$ represents the recurring decimal fraction 0.2008008008008... . When the answers to the following calculations are arranged in numerical order, which one is in the middle?

A $226 \div 1125 = 0.200\dot{8}$ B $251 \div 1250 = 0.2008$ C $497 \div 2475 = 0.20\dot{0}\dot{8}$ D $1003 \div 4995 = 0.2\dot{0}0\dot{8}$ E $2008 \div 9999 = 0.\dot{2}00\dot{8}$

18. C The five options are: A 0.20088888… B 0.20080000… C 0.20080808… D 0.20080080… E 0.20082008….
So in ascending order they are B D C E A.

19. Which of the following is equal to $(1 + x + y)^2 - (1 - x - y)^2$ for all values of x and y ?

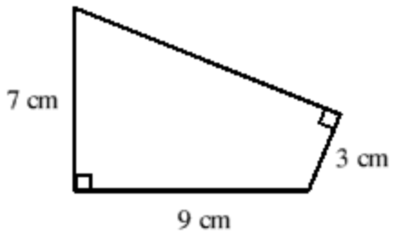
A $4x$ B $2(x^2 + y^2)$ C 0 D $4xy$ E $4(x + y)$

19. E Using 'the difference of two squares':

$$(1+x+y)^2-(1-x-y)^2=(1+x+y+1-x-y)(1+x+y-1+x+y)=2(2x+2y)=4(x+y).$$

20. What, in cm^2, is the area of this quadrilateral?

A 48 B 50 C 52 D 54 E 56



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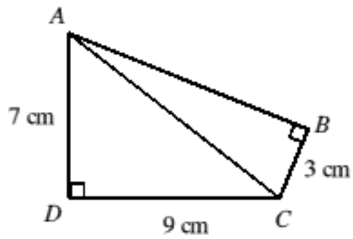
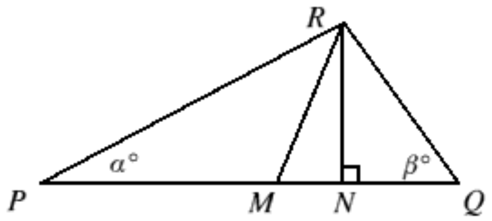
20. A Let the vertices of the quadrilateral be A, B, C, D as shown. Then, by Pythagoras' Theorem: $AC^2=AD^2+DC^2=(7^2+9^2)\text{ cm}^2=130\text{ cm}^2$. Similarly, $AB^2+BC^2=AC^2$, so $AB^2=(130-9)\text{ cm}^2=121\text{ cm}^2$. Therefore AB has length 11 cm and the area, in cm^2, of quadrilateral $ABCD$ is $\frac{1}{2}\times 9\times 7+\frac{1}{2}\times 3\times 11=48$.



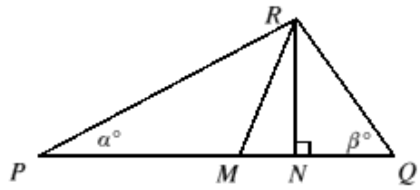
21. In triangle PQR, $\angle QPR = \alpha°$ and $\angle PQR = \beta°$, where $\alpha < \beta$. The line RM bisects $\angle PRQ$ and RN is the perpendicular from R to the line PQ. What is the size, in degrees, of $\angle MRN$?



A $\frac{180-(\alpha+\beta)}{2}$ B $\frac{\beta - \alpha}{2}$ C $\frac{\alpha + 2\beta}{2}$ D $\frac{360-\alpha-2\beta}{2}$ E $\frac{\alpha + \beta}{2}$

0831

21. B The sum of the interior angles of a triangle is 180°, so $\angle PRQ = (180 - \alpha - \beta)°$; hence $\angle QRM = \left(90 - \frac{\alpha}{2} - \frac{\beta}{2}\right)°$. As RN is perpendicular to PQ, $\angle NRQ = (90 - \beta)°$. So $\angle MRN = \left(90 - \frac{\alpha}{2} - \frac{\beta}{2}\right)° - (90 - \beta)° = \left(\frac{\beta}{2} - \frac{\alpha}{2}\right)°$.



22. At a cinema, a child's ticket costs £4.20 and an adult's ticket costs £7.70. When a group of adults and children went to see a film, the total cost was £C. Which of the following is a possible value of C?

A 91 B 92 C 93 D 94 E 95

0832

22. A Both £4.20 and £7.70 are multiples of 70p, so £C must also be a multiple of 70p. Of the options given, only £91 is a multiple of 70p, but it remains to check that a total cost of £91 is possible. If there are 7 children and 8 adults, then the total cost is $7 \times £4.20 + 8 \times £7.70 = £29.40 + £61.60 = £91$.

23. Beatrix has a 24-hour digital clock on a glass table-top next to her desk. When she looked at the clock at 13:08, she noticed that the reflected display also read 13:08, as shown.

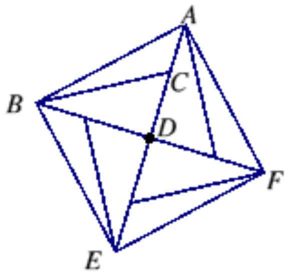
How many times in a 24-hour period do the display and its reflection give the same time?

A 12 B 36 C 48 D 72 E 96

23. E The only digits which will appear the same when reflected in the glass table-top are 0, 1, 3 and 8. So it is necessary to find the number of times in a 24-hour period that the display on the clock is made up only of some or all of these four digits. The first of the digits, therefore, may be 0 or 1; the second digit may be 0, 1, 3 or 8; the third digit may be 0, 1 or 3; the fourth digit may be 0, 1, 3 or 8. So the required number is $2 \times 4 \times 3 \times 4 = 96$.

24. The diagram has order 4 rotational symmetry about D. If angle ABC is 15° and the area of $ABEF$ is 24 cm^2, what, in cm, is the length of CD ?

A 1 B $\sqrt{3}$ C 2 D $\sqrt{5}$ E $2\sqrt{3} - 1$

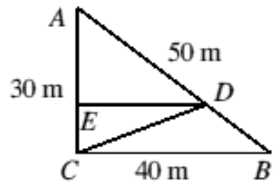


24. C As the figure has rotational symmetry of order 4, $ABEF$ is a square.
Area $ABEF = 4 \times$ area $\Delta BDA = 4 \times \frac{1}{2}BD \times DA = 2BD^2 = 24\,\text{cm}^2$ so $BD = \sqrt{12}$ cm $= 2\sqrt{3}$ cm. As $ABEF$ is a square, $\angle ABD = 45°$ so $\angle CBD = 45° - 15° = 30°$.
Therefore $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{CD}{BD} = \dfrac{CD}{2\sqrt{3}}$, so $CD = 2$ cm.

25. A garden has the shape of a right-angled triangle with sides of length 30, 40 and 50. A straight fence goes from the corner with the right-angle to a point on the opposite side, dividing the garden into two sections which have the same perimeter. How long is the fence?

A 25 B $8\sqrt{3}$ C $5\sqrt{11}$ D $5\sqrt{39}$ E $12\sqrt{5}$

25. E In the diagram on the right, triangle ABC represents the garden, CD represents the fence and E is the foot of the perpendicular from D to AC.



The two sections of the garden have the same perimeter so AD is 10 m longer than DB. Hence $AD = 30$ m and $DB = 20$ m. As $\angle AED$ and $\angle ACB$ are both right angles, triangles AED and ACB are similar.

So $\frac{AE}{AC} = \frac{AD}{AB} = \frac{30}{50}$. Hence $AE = \frac{3}{5} \times 30\,\text{m} = 18\,\text{m}$. So $EC = (30 - 18)\,\text{m} = 12\,\text{m}$.

Also, $\frac{ED}{CB} = \frac{AD}{AB} = \frac{30}{50}$. Hence $ED = \frac{3}{5} \times 40\text{ m} = 24\text{ m}$.

Finally, by Pythagoras' Theorem: $CD^2 = EC^2 + ED^2 = (12^2 + 24^2)\,\text{m}^2 = 5 \times 12^2\,\text{m}^2$.

So the length of the fence is $12\sqrt{5}\,\text{m}$.