Intermediate Mathematical Challenge 2007

1. At midnight on 15 December 2005, the moon reached its highest point in the sky, an event which occurs every 18.6 years. In which year will it next occur?

   A 2007 B 2008 C 2023 D 2024 E 2191

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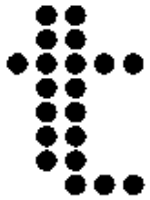
1. D The date 18 years after 15 December 2005 will be 15 December 2023. Since a further 0.6 years, i.e. just over 7 months, will then elapse, the moon will next reach its highest point in the sky in 2024.

2. The information display on a train shows letters by illuminating dots in a rectangular 5×8 array. In the letter 't' shown, what fraction of the dots in the array is illuminated?



A $\frac{9}{20}$ B $\frac{19}{40}$ C $\frac{1}{2}$ D $\frac{21}{40}$ E $\frac{11}{20}$

2. C There are forty dots in the 5×8 array and twenty of these are illuminated to form the letter 't'.

3. In how many ways can a square be cut in half using a single straight line cut?

A 1 B 2 C 4 D 8 E Infinitely many

3. **E** Any line which passes though the centre of the square divides the square into two congruent shapes. An example is shown on the right.

4. Between them, Ginger and Victoria eat two thirds of a cake. If Ginger eats one quarter of the cake, what fraction of the cake does Victoria eat?

A $\frac{1}{2}$ B $\frac{2}{5}$ C $\frac{3}{8}$ D $\frac{4}{9}$ E $\frac{5}{12}$

4. **E** The fraction of the cake which Victoria eats is $\frac{2}{3} - \frac{1}{4}$, that is $\frac{5}{12}$.

5. What is the value of $(12340 + 12{\cdot}34) \div 1234$?

A 100.01 B 100.1 C 10.001 D 10.01 E 10.1

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5. D $(12340 + 12.34) \div 1234 = 12340 \div 1234 + 12.34 \div 1234 = 10 + 0.01 = 10.01.$

6. The sum of 9 consecutive positive whole numbers is 2007. What is the difference between the largest and smallest of these numbers?

A 8 B 9 C 10 D 18 E 223

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6. A The difference between the largest and smallest of 9 consecutive whole numbers is 8, irrespective of the sum of the numbers. However in this case, since the average of the 9 numbers is 223, we can say they are 219, 220, …, 226, 227.

7. If the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are all multiplied together, how many zeros are at the end of the answer?

A 1 B 2 C 3 D 4 E 10

7. B The product = $1 \times 3 \times 4 \times 6 \times 7 \times 8 \times 9 \times (2 \times 5) \times 10$. The product of the first seven of these numbers is not a multiple of 10, so the original product is a multiple of 100, but not a multiple of 1000.

8. The mean of three numbers x, y and z is x. What is the mean of y and z?

A $\frac{1}{2}x$ B x C $2x$ D $3x$ E $4x$

8. B The sum of the three numbers is $3x$, so $y + z = 2x$. Hence the mean of y and z is x.

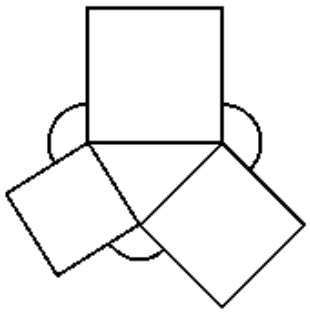
9. A male punky fish has 9 stripes and a female punky fish has 8 stripes. I count 86 stripes on the fish in my tank. What is the ratio of male fish to female fish?

A 2 : 3 B 3 : 2 C 4 : 1 D 4 : 7 E 7 : 4

9. B If there are f females and m males then $86 = 8f + 9m = 8(f + m) + m$. Now $86 = 8 \times 10 + 6 = 8 \times 9 + 14 = 8 \times 8 + 22\ldots$. Only the first of these is feasible (because $f + m \geqslant m$). So $m = 6$ and $f = 4$.

10. The diagram shows three squares drawn on the sides of a triangle. What is the sum of the three marked angles?

A 180° B 270° C 360°
D 450° E It depends on the shape of the triangle



10. C At each vertex of the triangle, four angles meet. In all, these twelve angles comprise six right angles, the three interior angles of the triangle and the three marked angles. So the sum of the marked angles $= 3 \times 360° - 6 \times 90° - 180° = 360°$.

11. The numbers 72, 8, 24, 10, 5, 45, 36, 15 are grouped in pairs so that the product of each pair is the same. Which number is paired with 10?

A 36 B 45 C 24 D 15 E 72

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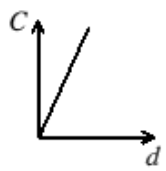
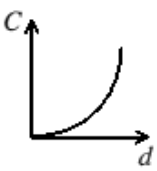
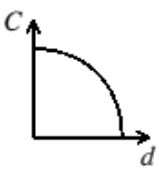
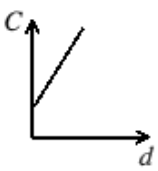
11. A As the product of each pair has the same value, this value must be product of the smallest and largest numbers, that is 5×72. So the number which is paired with 10 is $(5 \times 72) \div 10$, that is 36.

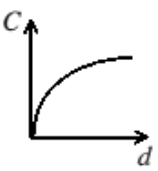
12. Which of the following could be the graph showing the circumference C of a circle in terms of its diameter d ?

A 

B 

C 

D 

E 

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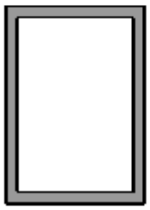
12. A The formula which gives C in terms of d is $C = \pi d$, so the graph of C versus d is a straight line which passes through the origin.

13. A 30 cm × 40 cm page of a book includes a 2 cm margin on each side, as shown.
What percentage of the page is occupied by the margins?

A 14% B 16% C 18% D 20% E 22%



13. E The area of the page inside the margins is 26 cm × 36 cm, that is 936 cm^2. So the percentage of the page occupied by the margins is $\frac{264}{1200} \times 100\%$, that is 22%.

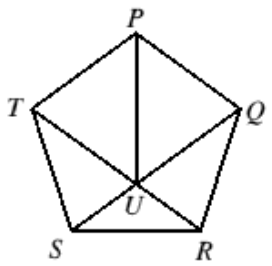
14. If p is a positive integer and q is a negative integer, which of the following is greatest?

A $p-q$ B $q-p$ C $p+q$ D $-p-q$ E More information needed

14. A Both p and $-q$ are positive numbers. Hence $p + (-q)$ is the largest of the alternatives.

15. The diagram shows a regular pentagon $PQRST$. The lines QS and RT meet at U. What is the size of angle PUR?

A 108° B 112° C 116° D 126° E 132°



15. D Each interior angle of a regular pentagon is 108°, so $\angle SRQ = 108°$. As $SR = QR$, triangle SRQ is isosceles with $\angle RQS = \angle RSQ = \angle 36°$. Similarly, $\angle SRT = \angle STR = 36°$. So $\angle SUR = (180 - 2 \times 36)° = 108°$.
From the symmetry of the figure, $\angle PUR = \angle PUS = (360° - 108°) \div 2 = 126°$.

16. A wooden cube with edge length 12 cm is cut into cubes with edge length 1 cm. What is the total length of all the edges of all these centimetre cubes?

A 12 cm B 12^2 cm C 12^3 cm D 12^4 cm E 12^5 cm

16. D In total, there are 12^3 cubes with edge length 1 cm. Each of these centimetre cubes has 12 edges, so the sum of the lengths of these edges is 12 cm. Therefore the total length of the edges of all the centimetre cubes is $12^3 \times 12$ cm, that is 12^4 cm.

17. Grannie's watch gains 30 minutes every hour, whilst Grandpa's watch loses 30 minutes every hour. At midnight, they both set their watches to the correct time of 12 o'clock. What is the correct time when their two watches next agree?

Grannie's watch: 

Grandpa's watch: 

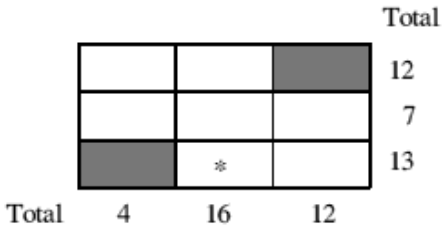
A 6 am B 9 am C 12 noon D 3 pm E 6 pm

17. C The two watches will next agree when Grannie's watch has gained twelve hours relative to Grandpa's watch. Each hour, Grannie's watch gains one hour relative to Grandpa's watch, so it will take 12 hours for this to happen. At this time, both watches will show a time of 6 o'clock.

18. One of the digits 1 to 9 is put in each unshaded square so that no digit is repeated and the totals of the entries in the rows and columns are as shown.

What number goes in the starred square?

A 1 B 3 C 5 D 7 E 9



18. C From the second row we see that c, d, e are 1, 2, 4 in some order; and from the third column we see that $e > 2$. Hence $e = 4$ and we may now deduce that $g = 8$ and so $f = 5$.
(*Although their values are not required, it is now also possible to deduce that* $c = 1$, $a = 3, b = 9, d = 2$.)

			Total
a	b		12
c	d	e	7
	f	g	13
Total	4	16	12

19. The following sequence continues indefinitely:

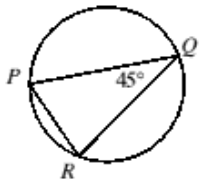
$27 = 3 \times 3 \times 3$, $207 = 3 \times 3 \times 23$, $2\,007 = 3 \times 3 \times 223$, $20\,007 = 3 \times 3 \times 2\,223, \ldots$

Which of the following integers is a multiple of 81?

A 200 007 B 20 000 007 C 2 000 000 007 D 200 000 000 007 E 20 000 000 000 007

19. E As each of the five given options is a member of the sequence, each is a multiple of 9. So we require the third factor of the number, that is the factor consisting of several 2s followed by a single 3, to be a multiple of 9 also. This is true if and only if the sum of its digits is a multiple of 9. For each number in the sequence, the number of 2s in its third factor equals the number of 0s in that number. The options given have 4, 6, 8, 10, 12 zeros, corresponding to their third factors having digital sums of 11, 15, 19, 23, 27 respectively. Of these, only 27 is a multiple of 9 so the correct answer is 20 000 000 000 007.

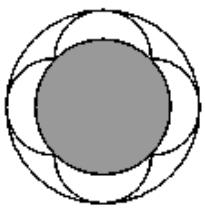
20. P, Q, R are points on the circumference of a circle of radius 4 cm. $\angle PQR = 45°$. What is the length of chord PR?



A 4 cm B $3\sqrt{3}$ cm C $4\sqrt{2}$ cm D 5 cm E 6 cm

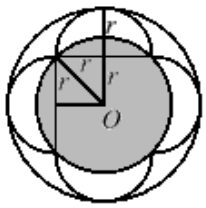
20. C Let O be the centre of the circle. Then $\angle POR = 90°$ as the angle subtended by an arc at the centre of a circle is twice the angle subtended by that arc at a point on the circumference of the circle. So triangle POR is an isosceles right-angled triangle with $PO = RO = 4$ cm. Let the length of PR be x cm. Then, by Pythagoras' Theorem, $x^2 = 4^2 + 4^2 = 2 \times 4^2$ and so $x = 4\sqrt{2}$.

21. The diagram shows two circles and four equal semi-circular arcs. The area of the inner shaded circle is 1. What is the area of the outer circle?



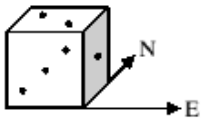
A $\sqrt{2}$ B 2 C $1+\sqrt{2}$ D $\frac{\pi}{2}$ E $\frac{9}{4}$

21. B From the symmetry of the figure, the two circles must be concentric. Let their centre be O. Let the radius of the semicircles be r. Then the radius of the outer circle is $2r$ and, by Pythagoras' Theorem, the radius of the inner shaded circle is $\sqrt{r^2 + r^2}$, that is $\sqrt{2}r$.



So the radii of the two circles are in the ratio $\sqrt{2} : 2$, that is $1 : \sqrt{2}$, and hence the ratio of their areas is $1 : 2$.

22. The diagram shows an ordinary die in which the scores on opposite faces always total 7. It is placed on a horizontal table with the '1' face facing East. The die is moved four times, rotating it each time through 90° about an edge. The faces in contact with the table are first 1, then 2, then 3, then 5. In which direction is the '1' face facing after this sequence of moves?



A West B East C North D South E Up

22. A The table shows the number on the face in contact with the table at the various stages described and also the numbers on the three faces of the die visible from the viewpoint in the question.

In contact with the table	5	1	2	3	5
Facing South	3	3	3	5	4
Facing East	1	2	6	6	6
Facing Up	2	6	5	4	2

As the '6' face is now facing East, the '1' face will be facing West.

23. As n takes each positive integer value in turn (that is, $n = 1$, $n = 2$, $n = 3$, and so on) how many different values are obtained for the remainder when n^2 is divided by $n + 4$?

A 1 B 8 C 9 D 16 E Infinitely many

23. C Note that

$$\frac{n^2}{n+4} = \frac{n^2-16}{n+4} + \frac{16}{n+4} = \frac{(n+4)(n-4)}{n+4} + \frac{16}{n+4} = n-4+\frac{16}{n+4}, (n \neq -4).$$

So when $n > 12$, the remainder when n^2 is divided by $n + 4$ is always 16. For $1 \leqslant n \leqslant 12$, the remainder when n^2 is divided by $n + 4$ is shown in the table below.

n	1	2	3	4	5	6	7	8	9	10	11	12
$n+4$	5	6	7	8	9	10	11	12	13	14	15	16
remainder	1	4	2	0	7	6	5	4	3	2	1	0

So there are 9 different remainders, namely 0, 1, 2, 3, 4, 5, 6, 7, 16.

24. In the diagram on the right, how many squares, of any size, are there whose entries add up to an even total?

A 12 B 20 C 32 D 36 E 45

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

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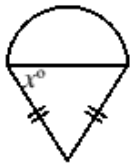
24. D Firstly, there are 12 unit squares which contain an even number. Every 2×2 square in the diagram has entries which consist of two odd numbers and two even numbers and hence have an even total. There are 16 of these. Each 3×3 square in the diagram, however, has entries which consist of five odd numbers and four even numbers (giving an odd total), or four odd numbers and five even numbers (giving an even total). There are 4 of the latter: those with 8, 12, 14 or 18 in the centre. Every 4×4 square in the diagram has entries which consist of eight odd numbers and eight even numbers and hence have an even total. There are 4 of these. Finally, the full 5×5 square contains 13 odd numbers and 12 even numbers, giving an odd total. So the required number is $12 + 16 + 4 + 4$, that is 36.

25. The diagram shows a semi-circle and an isosceles triangle which have equal areas. What is the value of $\tan x°$?



A 1 B $\frac{\sqrt{3}}{2}$ C $\frac{\pi}{\sqrt{3}}$ D $\frac{2}{\pi}$ E $\frac{\pi}{2}$

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25. E Let the radius of the semicircle be r and let the perpendicular height of the isosceles triangle be h. Then $\tan x° = \frac{h}{r}$.
Now the area of the semicircle $= \frac{1}{2}\pi r^2$, whilst the area of the triangle $= \frac{1}{2} \times 2r \times h = rh$. So $rh = \frac{1}{2}\pi r^2$, giving $\frac{h}{r} = \frac{\pi}{2}$.