Intermediate Mathematical Challenge 2005

1. If the following numbers are arranged in increasing order of size, which one is in the middle?

A 4.04 B 4.004 C 4.4 D 4.44 E 4.044

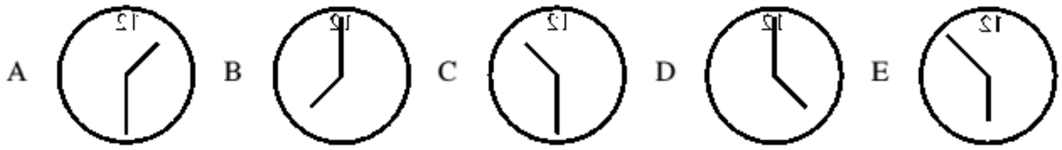
1. E In increasing order, the numbers are 4.004, 4.04, 4.044, 4.4, 4.44.

2. What is the difference between 10% of one million and 10% of one thousand?

A 9 900 B 9 990 C 90 900 D 99 900 E 999 900

2. D 10% of one million is 100 000; 10% of one thousand is 100; 100 000 – 100 = 99 900.

3. Professor Rosseforp has an unusual clock. The clock shows the correct time at noon, but the hands move anti-clockwise rather than clockwise. The clock is very accurate, however, so the hands move at the correct speeds. If you looked in a mirror at the Professor's clock at 1:30 pm, which of the following would you see?



3. A As the hands move anticlockwise, at 1:30 pm they would have the appearance they would normally have at 10:30. So, as the clock is viewed in a mirror, it would have the appearance shown in A.

4. Which of the following expressions is equal to 2005?

A $(1^2 + 1)(10^2 + 1)$ B $(2^2 + 1)(20^2 + 1)$ C $(3^2 + 1)(30^2 + 1)$ D $(4^2 + 1)(40^2 + 1)$ E $(5^2 + 1)(50^2 + 1)$

4. B The values of the expressions are, respectively, 202, 2005, 9010, 27 217 and 65 026.

5. How many of the statements in the box are true?

Any number which is divisible by 6 is even.	Any number which is divisible by 9 is odd.
The sum of any two odd numbers is even.	The sum of any two even numbers is odd.

A 0 B 1 C 2 D 3 E 4

5. **C** If a number is divisible by 6 then it must also be divisible by 2, so is even. 'The sum of any two odd numbers is even' is also true. However, not all multiples of 9 are odd (e.g. 18) and the sum of any two even numbers is even rather than odd, so two of the four statements are true.

6. A shop advertises 'Buy one, get one at half price'. For this offer, the average cost per item is the same as:

A Two for the price of one B Three for the price of one C Three for the price of two D Four for the price of three E Five for the price of four

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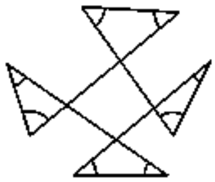
6. **D** The offer gives the purchaser two items for the price of one and a half items. So the average cost per item is the same as four items for the price of three.

7. In the diagram, what is the sum of the marked angles?

A 180° B 360° C 450° D 540° E 720°



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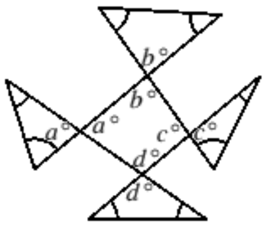
7. B Let the sizes of the interior angles of the quadrilateral in the centre of the figure be $a°$, $b°$, $c°$ and $d°$. Then $a + b + c + d = 360$. Each of these angles is vertically opposite to one of the unmarked angles in the four triangles, as shown. So the sum of the marked angles plus the four angles of the quadrilateral is equal to the sum of the interior angles of four triangles, that is 720°. Hence the sum of the marked angles is 720° – 360° = 360°.



8. What fraction of a 24-hour day does school take up, if school starts at 8:30am and finishes at 3:15pm?

A $\frac{9}{32}$ B $\frac{25}{96}$ C $\frac{13}{48}$ D $\frac{31}{96}$ E $\frac{18}{32}$

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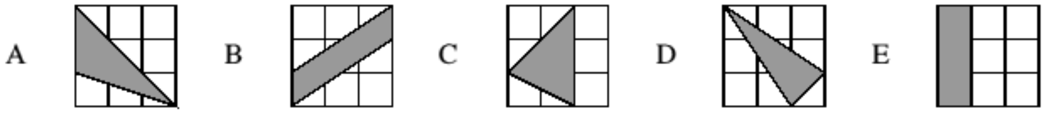
8. A The required fraction = $\frac{6\frac{3}{4}}{24} = \frac{27}{96} = \frac{9}{32}$.

9. Which of the following shaded regions has an area different from the other shaded regions?



9. D The area of triangle A = $\frac{1}{2} \times 2 \times 3 = 3$; the area of parallelogram B = $1 \times 3 = 3$; the area of triangle C = $\frac{1}{2} \times 3 \times 2 = 3$; the area of rectangle E = $1 \times 3 = 3$. However, the area of triangle D = $3 \times 3 - (3 + 3 + \frac{1}{2}) = 2\frac{1}{2}$.

10. Granny has taken up deep-sea fishing! Last week, she caught a fish so big that she had to cut it into three pieces (head, body and tail) in order to weigh it. The tail weighed 9kg and the head weighed the same as the tail plus one third of the body. The body weighed as much as the head and tail together. How much did the whole fish weigh?

A 18kg B 27kg C 54kg D 77kg E 84kg

10. C Let the weights in kg of the head and body of the fish be h and b respectively. Then $h = 9 + \frac{1}{3}b$ and $b = h + 9$. So $b = 9 + \frac{1}{3}b + 9$, that is $\frac{2}{3}b = 18$, which gives $b = 27$. Hence $h = 18$, so the whole fish weighed 54kg.

11. If two of the sides of a right-angled triangle are 5 cm and 6 cm long, how many possibilities are there for the length of the third side?

A 0 B 1 C 2 D 3 E 4

11. C The side of length 5cm cannot be the hypotenuse of the right-angled triangle as it is shorter than the side of length 6cm. If the 6cm side is the hypotenuse, then the third side of the triangle has length $\sqrt{11}$ cm. If the 6cm side is not the hypotenuse, then the hypotenuse has length $\sqrt{61}$ cm. These are the only two possibilities.

12. One gallon of honey provides enough fuel for a bee to fly about seven million miles. Roughly how many bees could fly one thousand miles if they had ten gallons of honey to share between them?

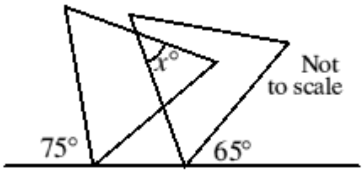
A 7 000 B 70 000 C 700 000 D 7 000 000 E 70 000 000

0522

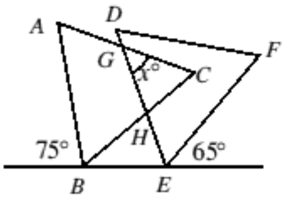
12. B Ten gallons of honey would provide enough fuel for one bee to fly about 70 000 000 miles. So the number of bees which could fly 1000 miles is approximately 70 000 000 $\div$ 1000, that is 70 000.

13. The diagram shows two equilateral triangles. What is the value of x?

A 70 B 60 C 50 D 40 E 30

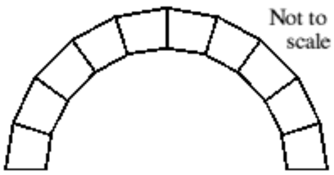


13. D Considering the angles at B and E:
$\angle CBE = (180 - 75 - 60)° = 45°$;
$\angle DEB = (180 - 65 - 60)° = 55°$. Therefore
$\angle GHB = (45 + 55)° = 100°$ (exterior angle theorem) and,
using the same theorem, $\angle HGC = (100 - 60)° = 40°$.

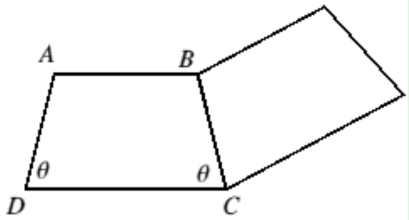


14. Ten stones, of identical shape and size, are used to make an arch, as shown in the diagram. Each stone has a cross-section in the shape of a trapezium with three equal sides. What is the size of the smallest angles of the trapezium?

A 72° B 75° C 81° D 83° E 85°



14. C Let $ABCD$ be the cross-section of one of the stones, as shown. As $AD = BC$, $ABCD$ is an isosceles trapezium with $\angle ADC = \angle BCD$ (the proof of which is left to the reader). If $\angle ADC = \angle BCD = \theta$, then 2θ is the interior angle of a regular 20-sided polygon, namely $(180 - 360/20)°$, which equals 162°. So θ is 81°.



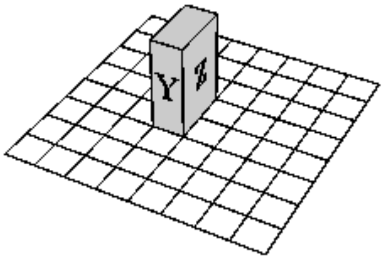
15. To make porridge, Goldilocks mixes together 3 bags of oats with 1 bag containing 20% wheat bran and 80% oats. All the bags have the same volume. What percentage of the volume of Goldilocks' porridge mixture is wheat bran?

A 5% B $6\frac{2}{3}$% C 20% D $26\frac{2}{3}$% E 60%

0525 ©UKMT

15. A Four bags of porridge contain one-fifth of a bag of wheatbran. So the proportion of wheatbran in the porridge is 1/20, that is 5%.

16. A $1 \times 2 \times 3$ block is placed on an 8×8 board, as shown with the 1×2 face 'X' at the bottom. It is rolled over an edge without slipping onto the 1×3 face Y, then onto the 2×3 face Z, then onto X, Y, Z again in that order. How many different squares on the board has the block occupied altogether, including the starting and ending positions?



A 18 B 19 C 20 D 21 E 22

16. B The two dark squares are those on which the block stands originally. After the first move, it occupies the squares labelled '1' or '1/5'. After the second move it occupies the squares labelled '2' and so on. After the fifth and final move, it occupies the squares labelled '5' or '1/5'.
As can be seen, 19 squares are occupied altogether.

	4	4	4	3	3		
	5	5	1/5	2	2		
	5	5	1/5	2	2		
			1	2	2		

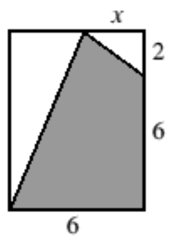
17. Platinum is a very rare metal, even rarer than gold. Its density is 21.45 g/cm^3. Assuming that the world production has been about 110 tonnes for each of the past 50 years, and negligible before that, which of the following has a comparable volume to that of the total amount of platinum ever produced?

A a shoe box B a cupboard C a house D Buckingham Palace E the Grand Canyon

17. C The volume of 1kg of platinum is $(1000/21.45)\text{cm}^3$, that is approximately 50cm^3. So 1 tonne of platinum has a volume of approximately $50\,000\text{cm}^3$, which is $1/20\text{m}^3$. The volume of platinum produced per year is therefore about 5m^3 and the total volume of platinum ever produced is approximately 250m^3. This is the volume of a cuboid measuring $10\text{m} \times 5\text{m} \times 5\text{m}$, which is comparable to a house.

18. Three-quarters of the area of the rectangle has been shaded. What is the value of x?

A 2 B 2.4 C 3 D 3.6 E 4



Not to scale

18. E The area of the rectangle is 48cm^2, so the unshaded area is 12cm^2. Therefore $\frac{1}{2} \times x \times 2 + \frac{1}{2} \times (6 - x) \times 8 = 12$, that is $x + 24 - 4x = 12$, so $x = 4$.

19. Trinni is fascinated by triangle numbers (1, 3, 6, 10, 15, 21, etc.) and recently, coming across a clock, she found that she could rearrange the twelve numbers 1, 2, 3, … 12 around the face so that each adjacent pair added up to a triangle number. She left the 12 in its usual place; what number did she put where the 6 would usually be?

A 1 B 4 C 5 D 10 E 11

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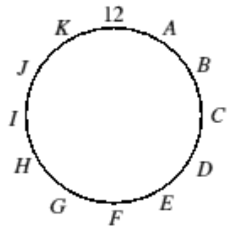
19. C For ease of reference, label the points $A, B, C, \ldots, K$ as shown. First note that $A = 3$ and $K = 9$ or vice versa. With no loss of generality, let $A = 3$. Then the only possible values for B, C and D are 7, 8 and 2 respectively. This gives $E = 1$ or 4 and, as $K = 9, J = 1$ or 6. If $J = 1$, then $E = 4$ and the only remaining possibilities for I, H and G are 5, 10 and 11 respectively. This means that $F = 6$, but $11 + 6$ is not a triangle number, so J is not 1 and must, therefore, be 6. This means that $I = 4$ and hence $E = 1$. The remaining values may now be assigned: $H = 11, G = 10$ and $F = 5$.



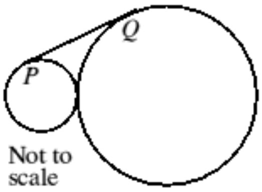
20. One of the following is the largest of nine consecutive positive integers whose sum is a perfect square. Which one is it?

A 118 B 128 C 138 D 148 E 158

0530

20. D The mean of the nine consecutive positive integers is the fifth of the numbers, so their sum is nine times the fifth number. As nine is itself a perfect square, the sum will be a perfect square if and only if the fifth number is a perfect square. For the options given, the fifth numbers are 114, 124, 134, 144 and 154 respectively.

21. Two circles with radii 1 cm and 4 cm touch. The point P is on the smaller circle, Q is on the larger circle and PQ is a tangent to both circles. What is the length of PQ?



Not to scale

A $\sqrt{17}$ cm B 3 cm C $2\sqrt{3}$ cm D $3\sqrt{2}$ cm E 4 cm

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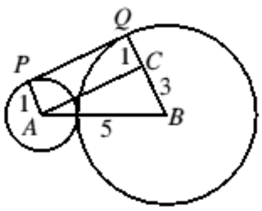
21. E The diagram shows points A and B, which are the centres of the two circles, and C, the point on BQ such that AC is parallel to PQ. As radii PA and QB are both perpendicular to tangent PQ, $APQC$ is a rectangle. So $\angle ACB$ is a right angle. The length of $AB = 1 + 4 = 5$; the length of $BC = 4 - 1 = 3$. So, by Pythagoras' Theorem, $AC = \sqrt{5^2 - 3^2} = 4$, which, therefore, is also the length of PQ.



22. Inspector Remorse had a difficult year in 2004. A crime wave in Camford meant that he had 20% more cases to solve than in 2003, but his success rate dropped. In 2003, he solved 80% of his cases, but in 2004 he solved only 60% of them. What was the percentage change in the number of cases he solved in 2004 compared with 2003?

A Down by 10% B Down by 8% C No change D Up by 8% E Up by 10%

0532

22. A Let the number of cases solved in 2003 be x. Then, as this was 80% of the number of cases, there were $5x/4$ cases to solve in 2003. So the number of cases to solve in 2004 was $5x/4 \times 6/5$, which is $3x/2$. Inspector Remorse solved 60% of these cases, that is $3x/2 \times 3/5$, which is $9x/10$. So the change in the number of cases solved was a 10% decrease.

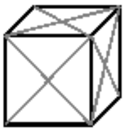
23. What is the area (in square units) of the triangle formed by the three lines whose equations are: $y - x = 6$, $x - 2y = 3$, $x + y = 6$?

A 55 B 60 C 65 D 70 E 75

0533

23. E The equations of the three lines must be considered in pairs to find the coordinates of their points of intersection, i.e. the coordinates of the vertices of the triangle. It is left to the reader to show that these are $(-15, -9)$, $(0, 6)$ and $(5, 1)$. The area of the triangle may now be found by enclosing it in a rectangle measuring 20×15 and subtracting the areas of the three surrounding triangles from that of the rectangle. This gives $300 - (112\frac{1}{2} + 12\frac{1}{2} + 100) = 75$.

24. The figure shows a cube of side 1 on which all twelve face diagonals have been drawn – creating a network with 14 vertices (the original eight corners, plus the six face centres) and 36 edges (the original twelve edges of the cube plus four extra edges on each face). What is the length of the shortest path along the edges of the network which passes through all 14 vertices?



A $1 + 6\sqrt{2}$ B $4 + 2\sqrt{2}$ C 6 D $8 + 6\sqrt{2}$ E $12 + 12\sqrt{2}$

24. A Figure (i) shows a net of the cube on which a possible path has been drawn, while figure (ii) shows a diagram of the cube on which the same path has been drawn. Each edge of the network which joins a corner to a face centre has length $1/\sqrt{2}$, while each edge which joins two adjacent corners has length 1. So the length of the path shown is $1 + 12 \times 1/\sqrt{2}$, that is $1 + 6\sqrt{2}$. This is the length of the shortest path along the edges of the network which passes through all 14 vertices. To prove this, we first note that to connect the 14 vertices we need a minimum of 13 edges, so the length of the shortest path must be at least $13 \times 1/\sqrt{2}$. A path of this length would move alternately between corners and face centres, but as there are 8 corners and 6 face centres this is impossible. At least one of the edges on the shortest path, therefore, must join two corners. So the length of the shortest path must be at least $1 + 12 \times 1/\sqrt{2}$. The diagrams show that such a path does exist so we are able to conclude that $1 + 6\sqrt{2}$ is indeed the length of the shortest path.

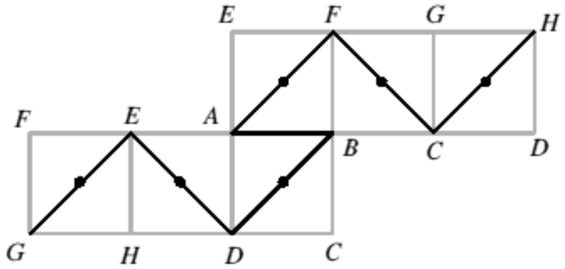


Figure (i)

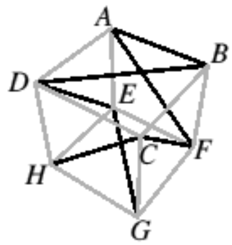
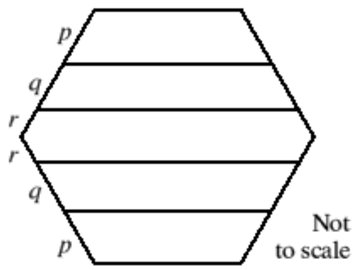


Figure (ii)

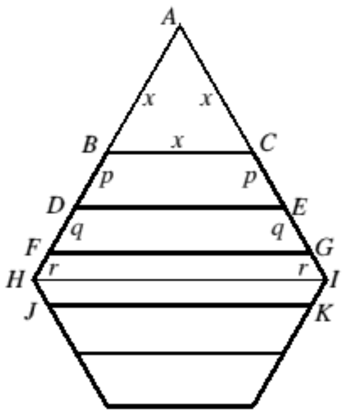
25. This regular hexagon has been divided into four trapezia and one hexagon. If each of the five sections has the same perimeter, what is the ratio of the lengths p, q and r ?

A 8 : 2 : 1 B 12 : 4 : 1 C 9 : 3 : 1

D 6 : 3 : 1 E 9 : 4 : 1



25. B Each exterior angle of a regular hexagon = $360° \div 6 = 60°$, so when sides HB and IC are produced to meet at A, an equilateral triangle, ABC, is created. Let the sides of this triangle be of length x. As BC, DE and FG are all parallel, triangles ABC, ADE and AFG are all equilateral. So $DE = DA = p + x$; $FG = FA = q + p + x$.



The perimeter of trapezium $BCED = x + p + x + 2p = 2x + 3p$; the perimeter of trapezium $DEGF = (p + x) + (q + p + x) + 2q = 2x + 2p + 3q$; the perimeter of hexagon $FGIKJH = 2((q + p + x) + 2r) = 2x + 2p + 2q + 4r$.

So $2x + 3p = 2x + 2p + 3q$; hence $p = 3q$. Also $2x + 2p + 3q = 2x + 2p + 2q + 4r$; hence $q = 4r$. So $p : q : r = 12r : 4r : r = 12 : 4 : 1$.