NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Friday, March 15th 1985

Time allowed - 3½ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. O is a point outside a circle. Two lines OAB, OCD through O meet the circle at A,B,C,D with A,C the midpoints of OB,OD respectively. Also the acute angle θ between the lines is equal to the acute angle at which each line cuts the circle. Find $\cos\theta$ and show that the tangents at A,D to the circle meet on the line BC .

2. A positive integer is called *evil* if the number of digits 1 in its binary expansion is even. For example $18 = (10010)_2$ is evil. Find the sum of the first 1985 evil positive integers.

3. Prove that the product of five consecutive positive integers is never a perfect square.

4. Delegations from 30 countries took part in a session of the International Mathematical Olympiad jury - the leaders and their deputies, 60 people in all. During the session some of the participants shook hands with each other but no leader shook hands with his deputy and no two people shook hands more than once. After the session the leader of the Mongolian team asked everybody how many times they had shaken hands. All the participants answered and the numbers they gave were all different. How many times did the Mongolian leader's deputy shake hands?

For full credit you must establish clearly that your answer is the only one consistent with the conditions.

5. ABCD is a tetrahedron which has a circumsphere passing through A, B, C, D and an in-sphere touching each triangular face at an interior point of that face. The two spheres have the same centre O . H is the orthocentre of triangle ABC and H' is the foot of the perpendicular from D on to the plane of that triangle.

Prove that AB = CD, AC = BD, AD = BC and that OH = OH' .