British Mathematical Olympiad

Round 2 : Tuesday, 27 February 2001

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge (probably 26-29 May). The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Washington DC, USA, 3-14 July) will then be chosen.

Do not turn over until **told to do so.**

2001 British Mathematical Olympiad
Round 2

1. Ahmed and Beth have respectively p and q marbles, with $p > q$.

   Starting with Ahmed, each in turn gives to the other as many marbles as the other already possesses. It is found that after $2n$ such transfers, Ahmed has q marbles and Beth has p marbles.
   Find $\dfrac{p}{q}$ in terms of n.

2. Find all pairs of integers (x, y) satisfying
$$1 + x^2y = x^2 + 2xy + 2x + y.$$

3. A triangle ABC has $\angle ACB > \angle ABC$.

   The internal bisector of $\angle BAC$ meets BC at D.

   The point E on AB is such that $\angle EDB = 90°$.

   The point F on AC is such that $\angle BED = \angle DEF$.

   Show that $\angle BAD = \angle FDC$.

4. N dwarfs of heights $1, 2, 3, \ldots, N$ are arranged in a circle. For each pair of neighbouring dwarfs the positive difference between the heights is calculated; the sum of these N differences is called the "total variance" V of the arrangement.

   Find (with proof) the maximum and minimum possible values of V.