British Mathematical Olympiad

Round 2 : Thursday, 16 February 1995

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (30 March – 2 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Toronto, Canada, 13–23 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 2–6 July before leaving for Canada.

Do not turn over until **told to do so.**

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1. Find all triples of positive integers (a, b, c) such that
$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

2. Let ABC be a triangle, and D, E, F be the midpoints of BC, CA, AB, respectively.

   Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ca = 9$.

   Prove that $0 < a < 1 < b < 3 < c < 4$.

4. (a) Determine, with careful explanation, how many ways $2n$ people can be paired off to form n teams of 2.

   (b) Prove that $\{(mn)!\}^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.