British Mathematical Olympiad

Round 2 : Thursday 13th February 1992

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

Before the end of February, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 2-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team for this summer's International Mathematical Olympiad (to be held in Moscow, 10-21 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Moscow.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Let p be an odd prime number. Prove that there are unique positive integers x, y such that $x^2 = y(y+p)$, and give the formulae for x and y in terms of p.

2. Let a, b, c, d be positive real numbers. Prove that
$$\frac{12}{a+b+c+d} \leq \frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{a+d}+\frac{1}{b+c}+\frac{1}{b+d}+\frac{1}{c+d} \leq \frac{3}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

3. The circumcircle of the triangle ABC has a radius R satisfying
$$AB^2 + AC^2 = BC^2 - R^2.$$
Prove that the angles of the triangle are uniquely determined, and state the values for the angles.

4. Dwarfish social life in the *Land Under the Mountain* is based on 'visiting' which has its own dwarfish rules. Two dwarves may visit each other only if they are acquainted, but there are three levels of friendship.

   Doorstep friends may visit each other and talk for as long as they like, but must remain on the doorstep, never crossing the threshold.

   Tea friends may cross each others' threshold, and even share a cup of tea, but must never stay for supper.

   Supper friends may visit each other and stay as long as they like, sharing ale, supper and a fireside chat.

   Each dwarf has exactly one friend of each kind, and the community is structured so that every pair A, B of dwarves is linked by a chain of acquaintances of various kinds so that A knows someone, who knows someone, who knows someone, ..., who knows B.

   (i) Prove that the number of dwarves must be an even number exceeding 2, and that, corresponding to each even $n > 2$, there may be such a community of n dwarves.

   (ii) The evil orcs want to isolate one group of dwarves from the rest of the community. To do this they must destroy some set of friendships. Suppose that they were to succeed in this by destroying

   F_d doorstep friendships, F_t tea friendships, F_s supper friendships,

   and that all of these friendships must be destroyed to isolate that group. Prove that F_d, F_t, F_s would have to be either all even or all odd.