UK Mathematical Olympiad for Girls

EGMO | 2012
European Girls' Mathematical Olympiad

23 June 2011

Instructions

Supported by
MILLFIELD

- *Time allowed: 3 hours.*
- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top left hand corner.*
- *To accommodate candidates sitting in other timezones, please do not discuss the paper on the internet until 8am BST on Friday 24 June.*

Do not turn over until **told to do so**.

A partnership of

UKMT

United Kingdom Mathematics Trust

Murray Edwards College
founded as New Hall

UK Mathematical Olympiad for Girls

23 June 2011

1. Three circles MNP, NLP, LMP have a common point P. A point A is chosen on circle MNP (other than M, N or P). AN meets circle NLP at B and AM meets circle LMP at C. Prove that BC passes through L.

   Many diagrams are possible. You need only solve this problem for one of the possible configurations.

2. The number 12 may be factored into three positive integers in exactly eighteen ways, these factorizations include $1 \times 3 \times 4$, $2 \times 2 \times 3$ and $2 \times 3 \times 2$. Let N be the number of seconds in a week. In how many ways can N be factored into three positive integers?

   A numerical answer is not sufficient. The calculation should be explained and justified.

3. Consider a convex quadrilateral and its two diagonals. These form four triangles.

   (a) Suppose that the sum of the areas of a pair of opposite triangles is half the area of the quadrilateral. Prove that at least one of the two diagonals divides the quadrilateral into two parts of equal area.

   (b) Suppose that at least one of the two diagonals divides the quadrilateral into two parts of equal area. Prove that the sum of the areas of a pair of opposite triangles is half the area of the quadrilateral.

4. Find a cubic polynomial $f(X)$ with integer coefficients such that whenever a, b, c are real numbers such that $a+b+c = 2$ and $a^2+b^2+c^2 = 2$, we have $f(a) = f(b) = f(c)$.

5. Let a be an even integer. Show that there are infinitely many integers b with the property that there is a unique prime number of the form $u^2 + au + b$ with u an integer.

   Note that an integer p is "prime" when it is positive, and it is divisible by exactly two positive integers. Therefore 1 is not prime, nor is -7.

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

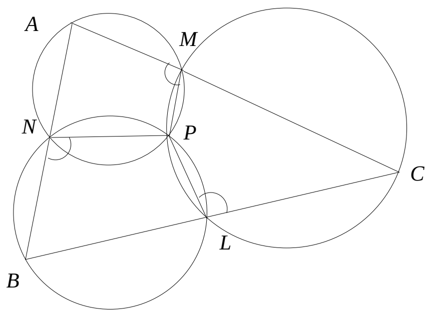
The mark allocation on Maths Olympiad papers is different from what you are used to at school. To get any marks, you need to make significant progress towards the solution. So 3 marks roughly means that you had most of the relevant ideas, but were not able to link them into a coherent proof. 8 or 9 marks means that you have solved the problem, but have made a minor calculation error or have not explained your reasoning clearly enough.

The authors of the problems include Christopher Bradley (Q1) and Geoff Smith (Q2, Q3, Q5); problem 4 was kindly supplied by David Monk, but, like many problems, its provenance is unclear. The UK MOG 2011 was marked on Sunday 3 July at the Holiday Inn, King's Cross by a team of Ceri Fiddes, Vesna Kadelburg, Joseph Myers, Vicky Neale, Geoff Smith and Alison Zhu, who also provided the remarks and extended solutions here.

1. Three circles MNP, NLP, LMP have a common point P. A point A is chosen on circle MNP (other than M, N or P). AN meets circle NLP at B and AM meets circle LMP at C. Prove that BC passes through L.

   Many diagrams are possible. You need only solve this problem for one of the possible configurations.

   Solution Here is a possible diagram. CMA and ANB are given to be straight lines, and we must show that BLC is a straight line.

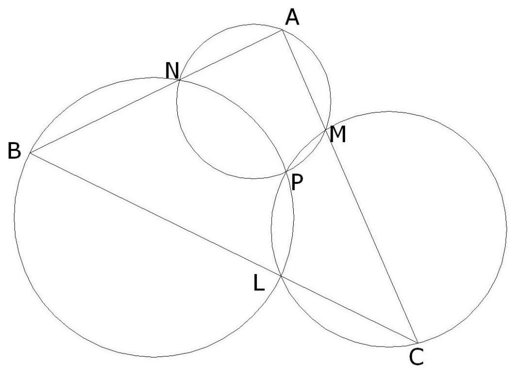
   The quadrilaterals $MANP$, $PNBL$ and $LCMP$ are all cyclic, so $\angle PLC = \angle PMA = \angle PNB$ by exterior angle to a cyclic quadrilateral. Now opposite angles of the cyclic quadrilateral $PNBL$ are supplementary, so angles BLP and $\angle PLC$ sum to a straight line, as required.

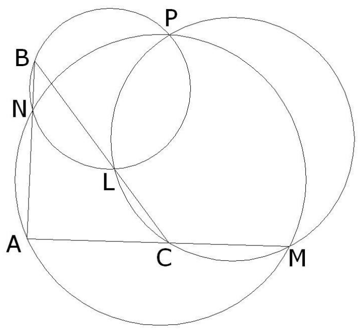
   Two angles are "supplementary" if they add to $180°$.

   Solution 2 Note that for fixed M, N and P, the sizes of $\angle MAN$, $\angle MBL$, $\angle LCN$ are invariant with respect to the position of A. We must also always have that $\angle MPN + \angle NPL + \angle LPM = 360°$. Therefore, as opposite angles in a cyclic quadrilateral add up to 180°, $\angle MAN + \angle NBL + \angle LCM = 3 \times 180° - 360° = 180°$. Since AMC and ANB are straight lines, the quadrilateral $ABLC$ is a triangle and we can now deduce that CLB is 180°.

Remarks Some candidates claimed that they had solved the problem by producing scale drawings or a couple of examples that confirmed that L lay on BC. However, this does not constitute a proof because this has not shown that the claim is true in all possible cases. Nevertheless, it is still important to draw accurate diagrams when trying to solve a question. Often, they can suggest the next step in your solution; you may spot that two triangles look similar and go on to prove this.

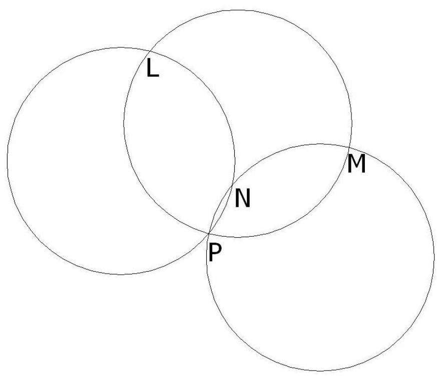
There was some confusion over what is meant by a "configuration". This usually refers to the relative positions of the three circles and the point A (such as the point A being on a minor as opposed to the major arc MN), and not to the exact sizes of the circles.





It turns out that for most special cases, one of the quickest solutions is to use circle theorems, as in our general solution. Some candidates solved the problem for a very specific case, which was given credit provided they made their assumptions completely clear. Unfortunately, most candidates who went down this route were not sufficiently careful about how they set up their configurations.

One problem arose from giving too few constraints. For example, some candidates stated that they were going to assume that the three circles had the same radii, but went on to claim that $MP = NP$. This requires an additional condition such as the circles being symmetric in the line PL. Here is an example where we have three congruent circles but $NP \neq MP$.



When writing out a proof, it is important to check very carefully that each statement follows from the previous statement. Try very hard to find counterexamples. So if you have written $A \Rightarrow B$, try to find a case where A is true and B is not true.

In other cases, students started off with a set of constraints leading to a configuration which might not exist. For example, it is not immediately obvious that we can have

P diametrically opposite B and C, with circles NPL and MPL of equal radii. The safest way of setting up a specific configuration would be to give a construction. Fix P. Specify M, N and L. Then construct A in relation to the existing points.

Returning to the question as it was meant to be solved, two key ideas were required. Firstly, we needed to connect lines NP, MP and LP and notice that we have the cyclic quadrilaterals $ANPM$, $NPLB$, $PMCL$. Secondly, we needed a strategy for showing that L lies on BC. One way is to show that $\angle BLP + \angle PLC = 180°$. Then BLC is a straight line.

Finally, be careful when labelling angles. Obviously in an accurate diagram, it is impossible not to draw L on BC, but for the purposes of this question, remember that we do not yet know that L is on BC. Therefore the statement "$NPLB$ is a cyclic quadrilateral and $\angle NPL = \alpha$, implying $\angle ABC = 180° - \alpha$" is not strictly correct. We only know that $\angle NBL = 180° - \alpha$.

2. The number 12 may be factored into three positive integers in exactly eighteen ways, these factorizations include $1 \times 3 \times 4$, $2 \times 2 \times 3$ and $2 \times 3 \times 2$. Let N be the number of seconds in a week. In how many ways can N be factored into three positive integers?

   A numerical answer is not sufficient. The calculation should be explained and justified.

   Solution For each prime number p, if $p^n \mid N$ but $p^{n+1} \nmid N$, then we can distribute the factors of p among the three factors in $\binom{n+2}{2}$ ways. To see this, imagine you have n stones. You divide them into three parts by adding two 'separator' stones to the pile, and then putting the stones in a row. You get a partition of n into three parts by looking at the number before the first separator, the number of stones between the separators, and the number of stones after the second separator. The number of such partitions is therefore given by the specified binomial formula. You then multiply over all primes (if you like you can restrict attention to those dividing N).

   In our case $N = 7 \times 24 \times 60^2 = 2^7 \times 3^3 \times 5^2 \times 7$ so the answer is

$$\tbinom{9}{2}\tbinom{5}{2}\tbinom{4}{2}\tbinom{3}{2} = 36 \times 10 \times 6 \times 3 = 6480.$$

   The notation $a \mid b$ means "a divides b". The notation $\binom{n}{r}$ is a binomial coefficient, sometimes also denoted ^nC_r.

   Remarks Most candidates correctly computed the value of N in this question ($N = 60 \times 60 \times 24 \times 7 = 604800$), and many realised that it would be helpful to find the prime factorization of N ($N = 2^7 \times 3^3 \times 5^2 \times 7$). The next step of the problem proved to be much harder. There were various attempts to try small cases and to look for patterns, which is of course a sensible strategy. It is important to remember that even if you notice a pattern, you still need to *justify* it carefully. Knowing that a pattern fits two or three cases does not prove that it works in general.

   A significant number of candidates tried to use the information given in the question about good factorizations of 12 ("good factorization" meaning the sort considered in the question). This information was in the question to illustrate that 1 is allowed in a factorization (e.g. $1 \times 3 \times 4$), and that the order of the factors is important ($2 \times 2 \times 3$ and $2 \times 3 \times 2$ count as different factorizations in this question). The candidates who tried to use the information about 12 typically tried to write N as the product of 12 and something else (or perhaps the product of 12^3 and something else) and then to multiply 18 by something. Unfortunately, this does not work. To see this, you could try counting the good factorizations of 12^2. There are not 18^2 of them. The problem is that the factors of 12 and 12 overlap, so we end up counting some factorizations too many times.

That thought might help to steer us in a useful direction: working with different primes separately. Our job is to count factorizations of N of the form $a \times b \times c$. We know that each of a, b and c must be a product of a power of 2, a power of 3, a power of 5 and a power of 7 (including 2^0 as a power of 2, and so on). We also need to use up all of the 2^7, 3^3, 5^2 and 7^1 that occur in the prime factorization of N. We could imagine that we have seven 2s, three 3s, two 5s and one 7 cut out from pieces of paper, and we need to assign them to three boxes, labelled a, b and c. We need to count the ways to do this.

Let us think about just the 2s. We have seven numbers, to put in three boxes. One way to do this is to imagine listing the seven 2s: 2 2 2 2 2 2 2. We can include two vertical bars, to separate the 2s for each of the three boxes. For example, the pattern 2 2 2 | 2 2 | 2 2 would correspond to three 2s in the a box, two 2s in the b box, and two 2s in the c box. Similarly, the pattern | 2 2 2 | 2 2 2 2 would correspond to no 2s in the a box, three 2s in the b box, and four 2s in the c box.

So we want to count the number of ways to list seven 2s and two vertical bars. We have nine spaces, and need to pick two of them for the vertical bars (the rest will automatically be filled by the seven 2s). In how many ways can we do this? You may have learned about binomial coefficients, in which case you will know that the answer is $\binom{9}{2}$ (sometimes also written as ${}^9C_2$). Alternatively, you can work it out directly. There are 9 places for the first vertical bar, and then 8 for the second, which gives 9×8 possibilities. But that counts each possibility twice (corresponding to swapping the vertical bars), so there are $\frac{9\times 8}{2} = 36$ possibilities. Fortunately, this is exactly the same as the binomial coefficient $\binom{9}{2}$.

That is the 2s dealt with. But we can now see how to deal with the 3s, 5s and 7. In general, if we have p^n (where p is a prime number and n is a natural number) then we want the binomial coefficient $\binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$. The argument in general is just the same as the argument above, thinking of including vertical bars and so on.

So there are $\frac{9\times 8}{2} = 36$ ways to put the 2s into the boxes, $\frac{5\times 4}{2} = 10$ ways to place the 3s, $\frac{4\times 3}{2} = 6$ ways to place the 5s, and just $\frac{3\times 2}{2} = 3$ ways to place the 7. (Of course, we can see the last of these immediately, without needing the argument above).

To complete the argument, we can notice that we can simply multiply these numbers (for each way to place the 2s, we have 10 ways to place the 3s, and so on). So the final answer is $36 \times 10 \times 6 \times 3 = 6480$.

We were pleased to see that some candidates managed to come up with a correct solution (along the lines of the one given here).

If you have not thought about it before, you might like to think about how to count the number of factors of $N = 604800$. Hint: do not try to list them all!

3. Consider a convex quadrilateral and its two diagonals. These form four triangles.

   (a) Suppose that the sum of the areas of a pair of opposite triangles is half the area of the quadrilateral. Prove that at least one of the two diagonals divides the quadrilateral into two parts of equal area.

   (b) Suppose that at least one of the two diagonals divides the quadrilateral into two parts of equal area. Prove that the sum of the areas of a pair of opposite triangles is half the area of the quadrilateral.

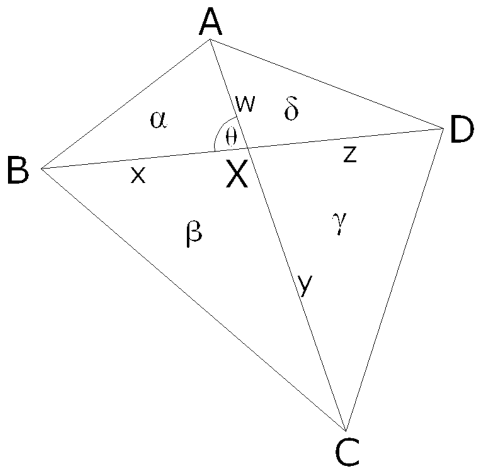
Solution Let the four triangles into which the quadrilateral $ABCD$ is broken by the diagonals have areas α, β, γ and δ respectively. For definiteness, let the diagonals meet at X and suppose that $[ABX] = \alpha$, $[BCX] = \beta$, $[CDX] = \gamma$ and $[DAX] = \delta$. Let $AX = w$, $BX = x$, $CX = y$ and $DX = z$.

Now if $\alpha + \gamma = \beta + \delta$, then $wx + zy = wz + xy$, so $wx + zy - wz - xy = 0$ and therefore $(w - y)(x - z) = 0$ so $w = y$ or $x = z$ so $\alpha + \delta = \beta + \gamma$ or $\alpha + \beta = \gamma + \delta$.

Conversely, suppose that $\alpha + \delta = \beta + \gamma$. Therefore $xw + wz = xy + yz$, so $w(x + z) = y(x + z)$ and therefore $w = y$ so $\alpha = \beta$ and $\gamma = \delta$. Therefore $\alpha + \gamma = \beta + \delta$.

Note that the area of a triangle is given by $\frac{1}{2}ab\sin C$, *and that* $\sin C = \sin(180° - C)$, *so the equations above result from cancellation of sine terms.*

Remarks The most common mistake made when attempting to answer this question was that candidates attempted to show that this is true for specific types of quadrilaterals, for example parallelograms or cyclic quadrilaterals. It is necessary for a correct argument to look at general quadrilaterals. Once you have drawn and labelled a quadrilateral for consideration you need to get a grip on the area in some way. The two most commonly used formulae for area of a triangle (as this is what we are dealing with in both parts) are $\frac{1}{2}bh$ and $\frac{1}{2}ab\sin C$, either will work.



In part (a) we assume that $\delta + \beta = \frac{1}{2}(\alpha + \beta + \gamma + \delta)$ (or $\alpha + \gamma = \frac{1}{2}(\alpha + \beta + \gamma + \delta)$ but this gives the same result). And it immediately follows that $\alpha + \gamma$ is also equal to half the area of the whole quadrilateral and so

$$\alpha + \gamma = \beta + \delta$$

and using $\frac{1}{2}ab\sin C$ this gives

$$\tfrac{1}{2}xw\sin\theta + \tfrac{1}{2}zy\sin\theta = \tfrac{1}{2}wz\sin(180° - \theta) + \tfrac{1}{2}xy\sin(180° - \theta).$$

It is clear to see that we should multiply by 2, and a little thought about $\sin\theta$ and $\sin(180° - \theta)$ should convince you they are the same thing and so they can also be factored out (N.B. we can divide through by $\sin\theta$ as it is definitely non-zero).

We are left with $xw + zy = wz + xy$ which rearranges to give

$$x(w - y) = z(w - y)$$

which means that either $x = z$ *or* $w - y = 0$. It is very important to notice both possibilities here, as it does not follow that x must be equal to z (consider $2 \times 0 = 3 \times 0$).

If $x = z$, then $wz + yz = wx + yx$, so

$$\tfrac{1}{2}wz\sin(180^\circ - \theta) + \tfrac{1}{2}yz\sin\theta = \tfrac{1}{2}wx\sin\theta + \tfrac{1}{2}yx\sin(180^\circ - \theta)$$

and we have $\delta + \alpha = \beta + \gamma$ which means the diagonal BD splits the area in half.

If $w = y$ the argument can run in exactly the same way to show that AC splits the quadrilateral in half.

The argument is easily reversible, and this is what was required in part (b).

4. Find a cubic polynomial $f(X)$ with integer coefficients such that whenever a, b, c are real numbers such that $a+b+c = 2$ and $a^2+b^2+c^2 = 2$, we have $f(a) = f(b) = f(c)$.

   Solution We have $2(ab+bc+ca) = (a+b+c)^2-(a^2+b^2+c^2) = 2$, so a, b, c are roots of $X^3 - 2X^2 + X - p$ where $p = abc$. Let $f(X)$ be $X(X-1)^2$ (i.e. $X^3 - 2X^2 + X$). Now $f(a) = f(b) = f(c) = abc$.

   Remarks Some candidates assumed that a, b and c had to be *integers*, although the question specified *real numbers*. Some candidates wrote the general cubic polynomial as $aX^3 + bX^2 + cX + d$; that notation was inappropriate for this question because the question already uses a, b and c as values that need not be the coefficients of the polynomial, and confusion arises if they are used with two different meanings.

   If $f(a) = f(b) = f(c) = k$, then a, b and c must be roots of $f(X) - k = 0$, so heuristically it seems appropriate to try letting $f(X) - k$ be the polynomial $(X-a)(X-b)(X-c)$ with those three roots. It is then necessary to notice that the non-constant terms of this polynomial have coefficients that do not depend on a, b and c, while $f(a) = f(b) = f(c)$ for any choice of the constant term.

   Another way of finding a candidate polynomial that some candidates used was substituting particular values of a, b and c satisfying the given conditions to obtain linear equations in the coefficients of f. This shows that f must have a particular form if it has the given property, and it is then necessary to show that the polynomial found does indeed have that property in order to complete the problem.

   Finding a possible f by trial and error with particular values of a, b and c, without any reason for it to work for *all* a, b and c satisfying the given conditions, was not rated highly.

5. Let a be an even integer. Show that there are infinitely many integers b with the property that there is a unique prime number of the form $u^2 + au + b$ with u an integer.

   Note that an integer p is "prime" when it is positive, and it is divisible by exactly two positive integers. Therefore 1 is not prime, nor is -7.

   Solution Note that a^2 is divisible by 4. As b ranges over the integers, every even perfect square arises as $a^2 - 4b$ (as do other less interesting values). The expression $\sqrt{a^2 - 4b}$ can take every even positive integer value. In particular, it can take the value $p - 1$ where p is any odd prime number (by choosing b appropriately).

   For such a choice of b, the roots α, β of the polynomial $X^2 + aX + b$ are integers which differ by $p - 1$. Let the larger root be β and let $w = \beta + 1$. Therefore $w^2 + aw + b = (w-\alpha)(w-\beta) = 1 \cdot p = p$. For integers $v > w$ we have $v^2 + av + b = (v-\alpha)(v-\beta)$ is a product of two integers, each greater than 1, so is not prime.

   For integers v in the range $\alpha \leq v \leq \beta$, the expression $v^2 + av + b$ is negative. For $v < \alpha$, the values obtained repeat those for $v > \beta$.

   Since there is a different choice of integer b for each odd prime number p, and there are infinitely many odd prime numbers, we are done.

Remarks No correct answer to this problem was obtained during the UK MOG.

We could start by trying to find just one b with the desired property, since it is not immediately clear that there is even one, never mind infinitely many. (We might try the problem with some particular numerical values of a first, to get a feel for what is going on, but we will move straight to the more general solution here.)

Our goal is to find some integer b so that there is a unique prime of the form $u^2 + au + b$ with u an integer. We might need to remember that there could be multiple (well, two) values of u that give the same value of $u^2 + au + b$.

We want to make sure that for almost all values of u, the quadratic u^2+au+b is not prime. What is a good way to make sure that a number N is not prime? Well, if we knew that we could factorize N into a product of two integers, neither of which is 1, then we would know that N is not prime.

This suggests that we could try to choose the integer b so that the polynomial $f(X) \equiv X^2 + aX + b$ has two integer roots, say α and β. Then $f(X)$ will factorize as $(X - \alpha)(X - \beta)$. When you evaluate $f(X)$ at u, this yields a factorization of $f(u)$. Provided that neither $|u - \alpha|$ nor $|u - \beta| = 1$, this will prevent $f(u)$ from being prime.

The value of a is given, and it is an even integer. The roots of $f(X)$ are

$$\frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

and these roots can be made to be integers by selecting b so that $a^2 - 4b$ is a square. Notice that the average of the roots is $-a/2$, and you can not alter that with your choice of b. However, the difference of the roots is $\sqrt{a^2 - 4b}$ and by appropriate choice of b, you can arrange that this has any even value that you like.

We are also going to need to make sure that $f(u)$ really is prime for at least one value of u. How can we do this? We could pick a prime p and then make sure that $w - \alpha = p$ and $w - \beta = 1$ for some suitable value of w, so that $f(w) = p$ (which is certainly prime). In order for this to be possible, we need the difference between the roots β and α to be $p - 1$. As is often the case, it is more convenient to choose p to be an odd prime.

So choose an odd prime p. For the given fixed even integer a, we choose b so that the roots α, β are integers, with $\alpha < \beta$ and $\beta - \alpha = p - 1$.

Let $w = \beta + 1$. Therefore $w^2 + aw + b = (w - \alpha)(w - \beta) = p \cdot 1 = p$, so we know that the quadratic $u^2 + au + b$ is a prime number for at least one value of u.

We still need to check that this is the only prime value of $u^2 + au + b$.

If v is an integer with $v > \beta+1$, then $v^2+av+b = (v-\alpha)(v-\beta)$, and each of these factors is an integer greater than 1, so $v^2 + av + b = (v - \alpha)(v - \beta)$ is not positive and so is not prime.

If v is an integer with $\alpha \leq v \leq \beta$, then $v - \alpha$ is positive and $v - \beta$ is negative, so $v^2 + av + b = (v - \alpha)(v - \beta)$ is not positive and so is not prime.

If v is an integer with $v < \alpha$, then $v^2 + av + b = (v - \alpha)(v - \beta) = (\beta + \alpha - v - \alpha)(\beta + \alpha - v - \beta)$, so the quadratic takes the same value as at $\beta + \alpha - v$ (which is greater than β). So for $v < \alpha$ the values obtained repeat those for $v > \beta$.

Therefore this choice of b ensures that $f(u)$ assumes just one prime value. It takes this value twice, at $u = \alpha - 1$ and $u = \beta + 1$.

So we have managed to find one value of b that meets the requirements. How can we show that there are in fact infinitely many? Our choice of b was determined by the odd prime p that we chose. But there are infinitely many odd primes p, and each of them gives rise to a different value of b. So there are infinitely many values of b that have the desired property.

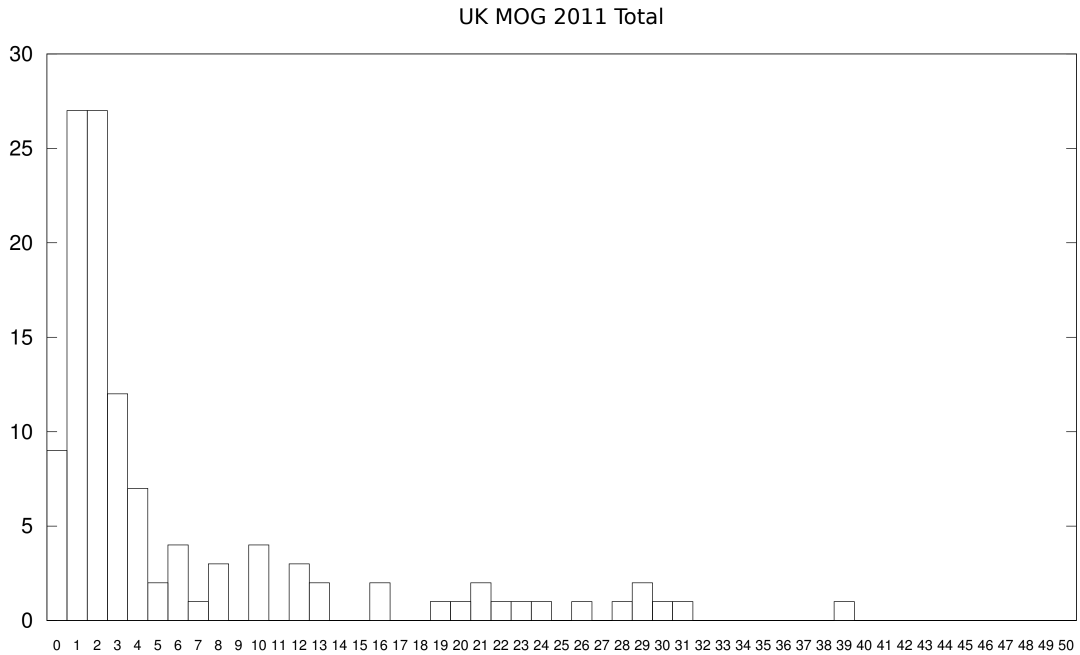
Above, we made sure that for almost all values of u, the quadratic $u^2 + au + b$ is not prime, by arranging for this polynomial to factorize into two linear factors. Might there be other ways to ensure this polynomial has few prime values? Two other ways that a polynomial could be almost never prime are that it could have only finitely many *positive* values—which cannot apply to this polynomial because the leading term has a positive coefficient—and that there could be some prime p such that it is always divisible by p. It turns out that this last case cannot apply to this polynomial (you might wish to think about why). And there is a conjecture (the Bunyakovskii conjecture) that these three reasons are the *only* reasons a polynomial with integer coefficients can fail to have infinitely many prime values. So any solution not involving making the polynomial factorize would imply a counterexample to this conjecture.

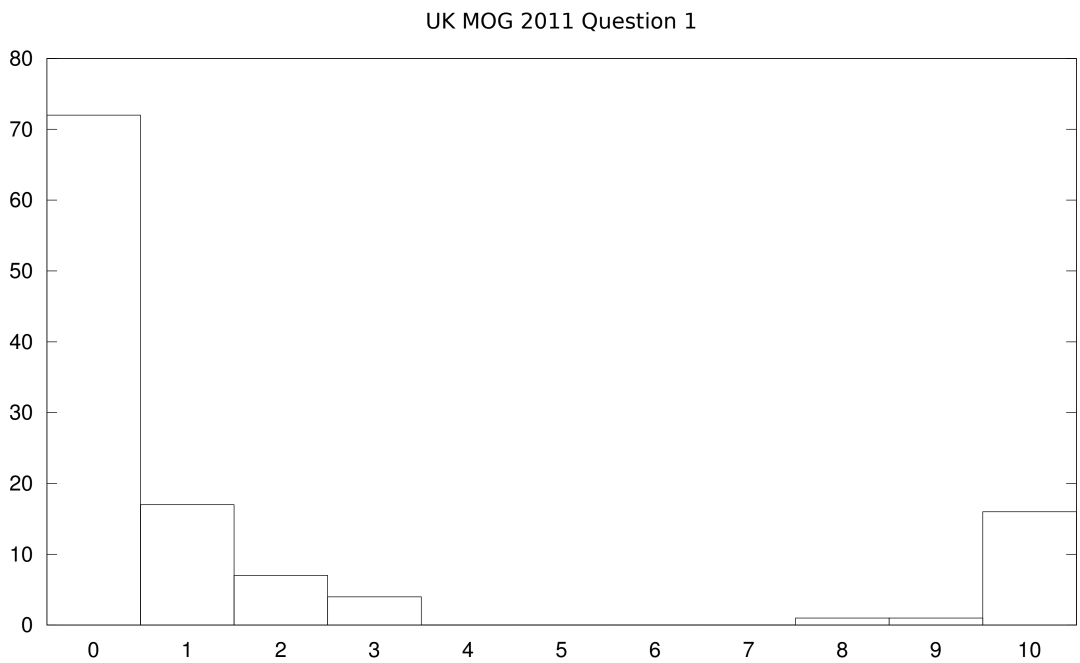
UK Mathematical Olympiad for Girls

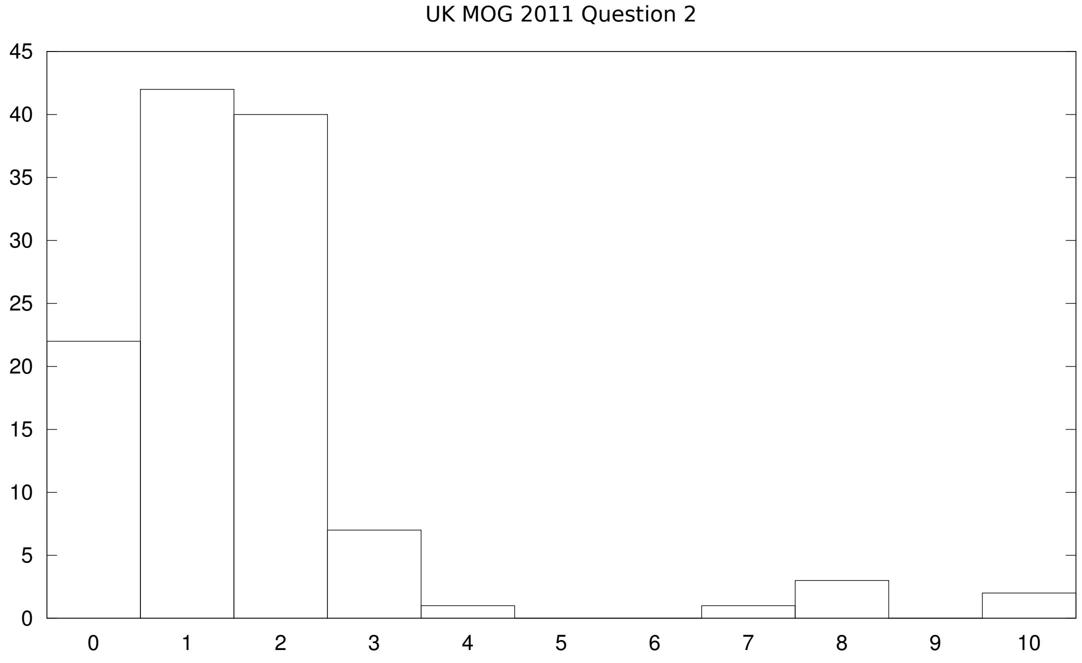
23 June 2011

Statistics of results

154 candidates entered UK MOG 2011, of whom 118 returned scripts. Statistics of the results of those 118 are shown below.






UK MOG 2011 Question 2
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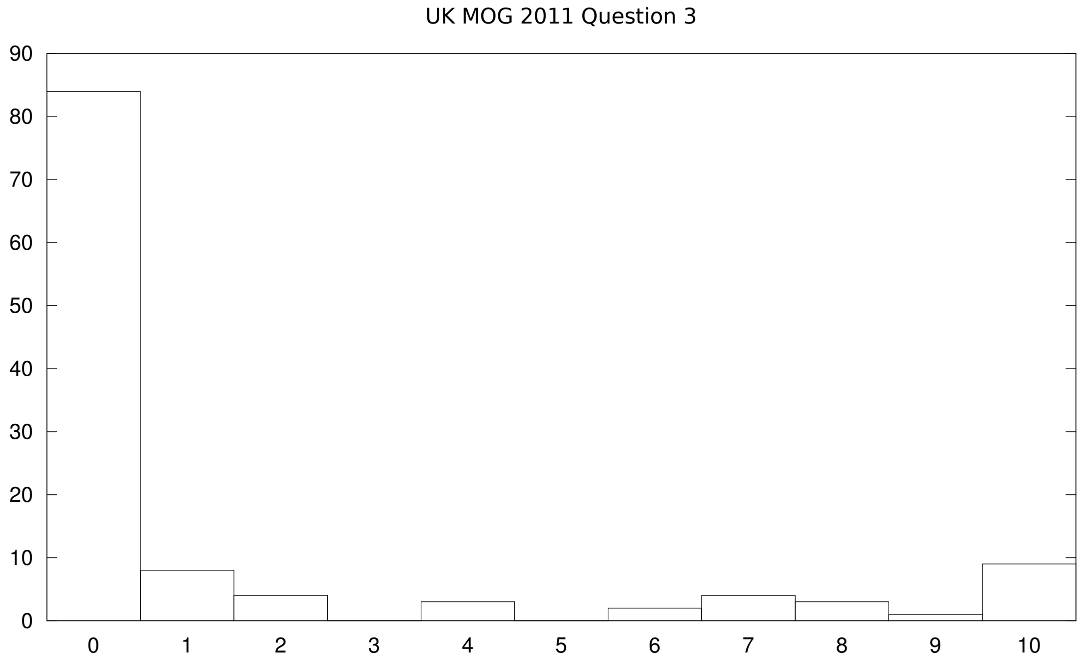
UK MOG 2011 Question 3
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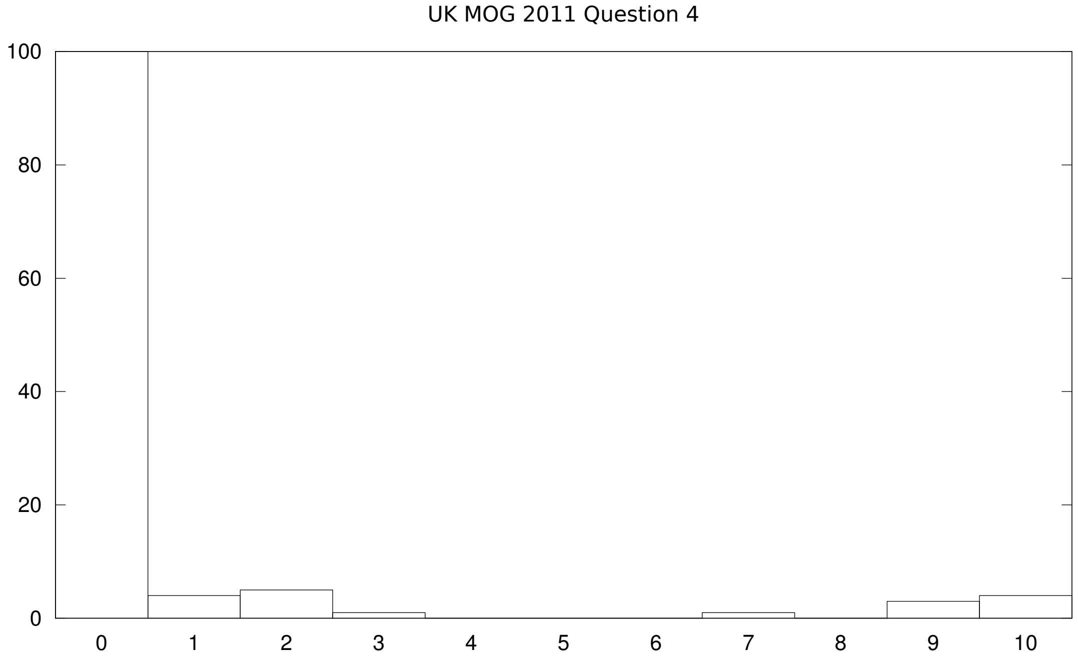
UK MOG 2011 Question 4
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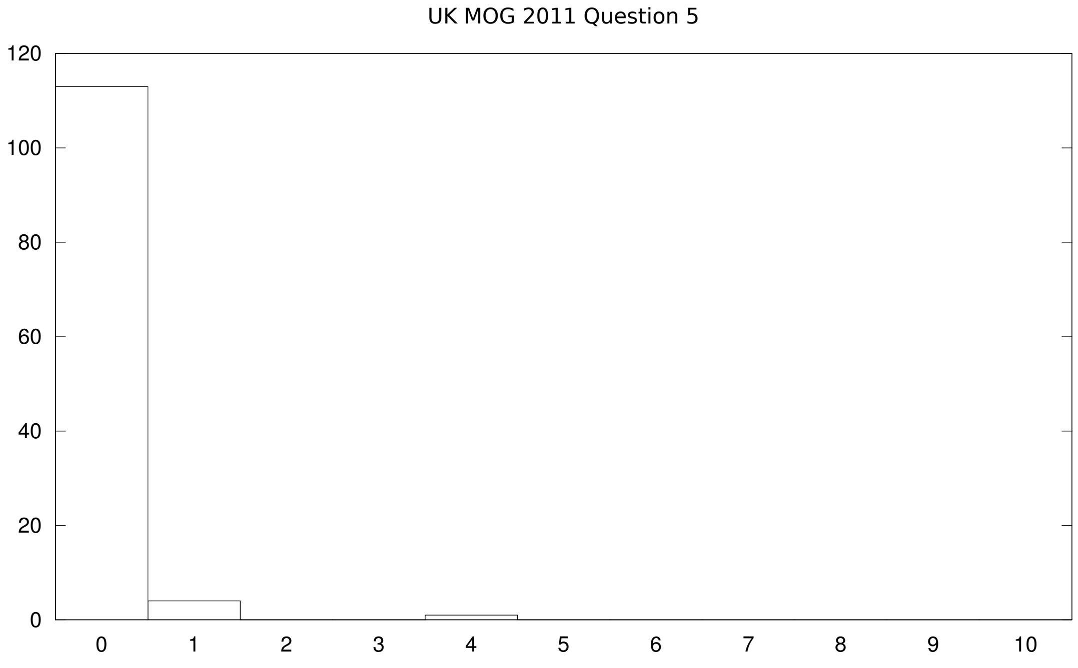
UK MOG 2011 Question 5
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