Section A: Pure Mathematics

1 Show that

$$\int_{\pi/6}^{\pi/4} \frac{1}{1-\cos 2\theta}\ \mathrm{d}\theta = \frac{\sqrt{3}}{2} - \frac{1}{2}\ .$$

By using the substitution $x = \sin 2\theta$, or otherwise, show that

$$\int_{\sqrt{3}/2}^{1} \frac{1}{1-\sqrt{(1-x^2)}}\ \mathrm{d}x = \sqrt{3} - 1 - \frac{\pi}{6}\ .$$

Hence evaluate the integral

$$\int_{1}^{2/\sqrt{3}} \frac{1}{y(y-\sqrt{(y^2-1^2)})}\ \mathrm{d}y\ .$$

2 Show that setting $z - z^{-1} = w$ in the quartic equation

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$

results in the quadratic equation $w^2 + 5w + 6 = 0$. Hence solve the above quartic equation.

Solve similarly the equation

$$2z^8 - 3z^7 - 12z^6 + 12z^5 + 22z^4 - 12z^3 - 12z^2 + 3z + 2 = 0\ .$$

3 The nth Fermat number, F_n, is defined by

$$F_n = 2^{2^n} + 1\,, \qquad n = 0,\ 1,\ 2,\ \ldots\ ,$$

where 2^{2^n} means 2 raised to the power 2^n. Calculate F_0, F_1, F_2 and F_3. Show that, for $k=1$, $k=2$ and $k=3$,

$$F_0F_1\ldots F_{k-1} = F_k - 2\ . \qquad (*)$$

Prove, by induction, or otherwise, that $(*)$ holds for all $k \geqslant 1$. Deduce that no two Fermat numbers have a common factor greater than 1.

Hence show that there are infinitely many prime numbers.

4 Give a sketch to show that, if $\mathrm{f}(x) > 0$ for $p < x < q$, then $\int_p^q \mathrm{f}(x)\,\mathrm{d}x > 0$.

(i) By considering $\mathrm{f}(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $\mathrm{f}(x) = a\sin^2 x - b\sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b\ln(q/p) < a\left(\frac{1}{p} - \frac{1}{q}\right) + c(q - p)\,.$$

5 The numbers x_n, where $n = 0, 1, 2, \ldots$, satisfy

$$x_{n+1} = kx_n(1 - x_n)\,.$$

(i) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n.

(ii) Given that $x_0 = x_1 = x_2 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a.

(iii) Given instead that $x_0 = x_2 = x_4 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a.

6 The lines l_1, l_2 and l_3 lie in an inclined plane P and pass through a common point A. The line l_2 is a line of greatest slope in P. The line l_1 is perpendicular to l_3 and makes an acute angle α with l_2. The angles between the horizontal and l_1, l_2 and l_3 are $\pi/6$, β and $\pi/4$, respectively. Show that $\cos\alpha\sin\beta = \frac{1}{2}$ and find the value of $\sin\alpha\sin\beta$. Deduce that $\beta = \pi/3$.

The lines l_1 and l_3 are rotated in P about A so that l_1 and l_3 remain perpendicular to each other. The new acute angle between l_1 and l_2 is θ. The new angles which l_1 and l_3 make with the horizontal are ϕ and 2ϕ, respectively. Show that

$$\tan^2\theta = \frac{3 + \sqrt{13}}{2}\,.$$

7 In 3-dimensional space, the lines m_1 and m_2 pass through the origin and have directions $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$, respectively. Find the directions of the two lines m_3 and m_4 that pass through the origin and make angles of $\pi/4$ with both m_1 and m_2. Find also the cosine of the acute angle between m_3 and m_4.

The points A and B lie on m_1 and m_2 respectively, and are each at distance $\lambda\sqrt{2}$ units from O. The points P and Q lie on m_3 and m_4 respectively, and are each at distance 1 unit from O. If all the coordinates (with respect to axes $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) of A, B, P and Q are non-negative, prove that:

(i) there are only two values of λ for which AQ is perpendicular to BP;

(ii) there are no non-zero values of λ for which AQ and BP intersect.

8 Find y in terms of x, given that:

$$\text{for } x < 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -y \quad \text{and} \quad y = a \text{ when } x = -1\,;$$

$$\text{for } x > 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y \quad \text{and} \quad y = b \text{ when } x = 1\,.$$

Sketch a solution curve. Determine the condition on a and b for the solution curve to be continuous (that is, for there to be no 'jump' in the value of y) at $x = 0$.

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = |\,\mathrm{e}^{x} - 1|\,y$$

given that $y = \mathrm{e}^{\,\mathrm{e}}$ when $x = 1$ and that y is continuous at $x = 0$. Write down the following limits:

$$\text{(i)} \quad \lim_{x\to+\infty} y\exp(-\,\mathrm{e}^{x})\,; \qquad \text{(ii)} \lim_{x\to-\infty} y\,\mathrm{e}^{-x}\,.$$

Section B: Mechanics

9 A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h, at distances a and b from O. Given that the particle just passes over the walls, find $\tan\alpha$ in terms of a, b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab} \,.$$

The heights of the walls are now increased by the same small positive amount δh. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2\alpha\,\delta\alpha \approx \frac{a+b}{ab}\,\delta h\,,$$

and deduce that $\delta\alpha > 0$. Show also that δV is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

10 A competitor in a Marathon of $42\frac{3}{8}$ km runs the first t hours of the race at a constant speed of 13 km h^{-1} and the remainder at a constant speed of $14 + 2t/T$ km h^{-1}, where T hours is her time for the race. Show that the minimum possible value of T over all possible values of t is 3.

The speed of another competitor decreases linearly with respect to time from 16 km h^{-1} at the start of the race. If both of these competitors have a run time of 3 hours, find the maximum distance between them at any stage of the race.

11 A rigid straight beam AB has length l and weight W. Its weight per unit length at a distance x from B is $\alpha W l^{-1}(x/l)^{\alpha-1}$, where α is a positive constant. Show that the centre of mass of the beam is at a distance $\alpha l/(\alpha+1)$ from B.

The beam is placed with the end A on a rough horizontal floor and the end B resting against a rough vertical wall. The beam is in a vertical plane at right angles to the plane of the wall and makes an angle of θ with the floor. The coefficient of friction between the floor and the beam is μ and the coefficient of friction between the wall and the beam is also μ. Show that, if the equilibrium is limiting at both A and B, then

$$\tan\theta = \frac{1-\alpha\mu^2}{(1+\alpha)\mu}\,.$$

Given that $\alpha = 3/2$ and given also that the beam slides for any $\theta < \pi/4$ find the greatest possible value of μ.

Section C: Probability and Statistics

12 On K consecutive days each of L identical coins is thrown M times. For each coin, the probability of throwing a head in any one throw is p (where $0 < p < 1$). Show that the probability that on exactly k of these days more than l of the coins will each produce fewer than m heads can be approximated by

$$\binom{K}{k} q^k (1-q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h-2l-1}{2\sqrt{h}}\right), \qquad h = L\Phi\left(\frac{2m-1-2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(.)$ is the cumulative distribution function of a standard normal variate.

Would you expect this approximation to be accurate in the case $K=7$, $k=2$, $L=500$, $l=4$, $M=100$, $m=48$ and $p=0.6$?

13 Let $\mathrm{F}(x)$ be the cumulative distribution function of the random variable X, with $\mathrm{F}(a)=0$ and $\mathrm{F}(b)=1$, and let

$$\mathrm{G}(y) = \frac{\mathrm{F}(y)}{2-\mathrm{F}(y)} \, .$$

Show that $\mathrm{G}(a)=0$, $\mathrm{G}(b)=1$ and that $\mathrm{G}'(y) \geqslant 0$. Show also that

$$\frac{1}{2} \leqslant \frac{2}{(2-\mathrm{F}(y))^2} \leqslant 2 \, .$$

The random variable Y has cumulative distribution function $\mathrm{G}(y)$. Show that

$$\tfrac{1}{2}\,\mathrm{E}(X) \leqslant \mathrm{E}(Y) \leqslant 2\mathrm{E}(X) \, ,$$

and that

$$\mathrm{Var}\,(Y) \leqslant 2\,\mathrm{Var}\,(X) + \tfrac{7}{4}(\mathrm{E}(X))^2 \, .$$

14 A densely populated circular island is divided into N concentric regions R_1, R_2, $\ldots$, R_N, such that the inner and outer radii of R_n are $n-1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2 - n/N$, independently of the number of accidents in any other region.

Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X, that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2 - \frac{1}{6}\left(1 + \frac{1}{N}\right)\left(4 - \frac{1}{N}\right) .$$

[Note: $\sum\limits_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1)$.]

Show also that

$$\mathrm{P}(X = k) = \frac{\mathrm{e}^{-2}N^{-k-2}}{k!}\sum_{n=1}^{N}(2n-1)(2N-n)^k\,\mathrm{e}^{n/N} .$$

Suppose now that $N = 3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48 + 45\,\mathrm{e}^{1/3} + 25\,\mathrm{e}^{-1/3}} .$$