Admissions Testing Service

STEP Solutions 2016

Mathematics

STEP 9465/9470/9475

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Test

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STEP I 2016 Solutions

Question 1

As a starter to the paper, this is a straightforward question in terms of its early demands and involves little more than the need to sort out some clearly-signposted algebra. To begin with, it is clear that, whenever n is odd, the expression (x^n + 1) has (x + 1) as a factor (by the *Factor Theorem*), so that

$$q_n(x) = x^{2n} - x^{2n-1} + x^{2n-2} + \ldots - x^3 + x^2 - x + 1.$$

Examining the $p_n(x)$'s in turn, using the binomial theorem (and *Pascal's Triangle* for the coefficients), gives

$p_1(x) = \left(x^2 + 2x + 1\right) - 3x(1) = x^2 - x + 1$,

$p_2(x) = \left(x^4 + 4x^3 + 6x^2 + 4x + 1\right) - 5x(x^2 + x + 1) = x^4 - x^3 + x^2 - x + 1$, and

$p_3(x) = \left(x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1\right) - 7x(x^2 + x + 1)^2$.

Expanding $(x^2 + x + 1)^2$ is relatively straightforward, and it is relatively easy to obtain the required results.

There are several ways to demonstrate that two given expressions of the given kind are not identically equal. One is to expand them both as polynomials and show that they are not the same. In this case,

$p_4(x) = x^8 - x^7 + x^6 + 2x^5 + 7x^4 + 2x^3 + x^2 - x + 1$ while $q_4(x) = x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$.

Alternatively, one need only show that one corresponding pair of coefficients are not the same – here, the coefficients of (say) x^5 are not equal. However, the simplest thing is to find any one value of x for which the two expressions give different values. It turns out, in fact, that only x = 0 actually does give equal outputs, so almost any chosen value of x would suffice, and the key is then to choose one for which the working involves the minimum of effort, such as $p_4(1) = 2^8 - 9.1.3^3 = 13 \neq q_4(1) = \dfrac{1^9 + 1}{1 + 1} = 1$.

In (ii) (a), the given numerical expression is clearly that for $q_1(x)$ with x = 300. Since p and q are the same thing when n = 1, we instead examine $p_1(300)$, and it becomes clear that if x is 3 times a perfect square (in this case 3×10^2) then we can use the difference-of-two-squares factorisation on $(301)^2 - (3\times10)^2$ to get the answer 271×331.

Part (b) has a similar thing going on, but here we need x to be 7 times a perfect square, and we find that we have $\left[\left(7^7 + 1\right)^3\right]^2 - 7^8\left(7^{14} + 7^7 + 1\right)^2$, which again requires the use of the difference-of-two-squares factorisation and yields $\left[\left(7^7 + 1\right)^3 - \left(7^{18} + 7^{11} + 7^4\right)\right] \times \left[\left(7^7 + 1\right)^3 + \left(7^{18} + 7^{11} + 7^4\right)\right]$ or

$$\left(7^{21} + 3.7^{14} + 3.7^7 + 1 - 7^{18} - 7^{11} - 7^4\right) \times \left(7^{21} + 3.7^{14} + 3.7^7 + 1 + 7^{18} + 7^{11} + 7^4\right),$$

either of which answers would suffice.

Question 2

Once again, this question begins with a very simple instruction, to differentiate a function of x, and will clearly involve the use of both the *Product Rule* (twice) and the *Chain Rule* (in order to deal with the log. term and the square-root). In principle, this looks very straightforward, though the key is to be careful not to overlook some aspect of the various processes at play, and then to simplify the resulting expressions in a suitable way. To begin with, one will obtain something that looks quite messy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(ax^2 + bx + c\right)\frac{1}{x+\sqrt{1+x^2}} \times \left(1 + \tfrac{1}{2}\left[1+x^2\right]^{-\frac{1}{2}}.2x\right) + \left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right)$$
$$+ \left(dx+e\right)\left(\tfrac{1}{2}\left[1+x^2\right]^{-\frac{1}{2}}.2x\right) + d\sqrt{1+x^2}$$

and it is easy to be put off; it is especially important not to attempt too much “in your head”. You should find this simplifies to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax^2+bx+c}{\left[x+\sqrt{1+x^2}\right]} \times \frac{\left[\sqrt{1+x^2}+x\right]}{\sqrt{1+x^2}} + \left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right) + \frac{x\left(dx+e\right)}{\sqrt{1+x^2}} + d\sqrt{1+x^2}$$

and collecting up terms suitably, and noting that $\frac{1}{\sqrt{1+x^2}} \equiv \frac{\sqrt{1+x^2}}{1+x^2}$ leads to an expression which contains only simple multiples of $\frac{1}{\sqrt{1+x^2}}$ and $\ln\left(x+\sqrt{1+x^2}\right)$; namely

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(a+2d)x^2+(b+e)x+(c+d)}{\sqrt{1+x^2}} + \left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right).$$

All the results of the remaining parts of the question can now be deduced by choosing suitable values for the constants a to e.

In (i), choosing $a = d = 0$, $b = 1$, $e = -1$ and $c = 0$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(0)x^2+(0)x+(0)}{\sqrt{1+x^2}} + \left(0+1\right)\ln\left(x+\sqrt{1+x^2}\right)$, so that $\int \ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x = x\ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2}$ (+ C).

In (ii), choosing $a = b = e = 0$ and $c = d = \frac{1}{2}$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(0+1)x^2+(0)x+(1)}{\sqrt{1+x^2}} + \left(0+0\right)\ln\left(x+\sqrt{1+x^2}\right)$, so that $\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + \frac{1}{2}x\sqrt{1+x^2}$ (+ C) .

And in (iii), choosing $a = \frac{1}{2}$, $b = e = 0$, $c = \frac{1}{4}$ and $d = -\frac{1}{4}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\frac{1}{2}-\frac{1}{2})x^2+(0)x+(\frac{1}{4}-\frac{1}{4})}{\sqrt{1+x^2}} + \left(x+0\right)\ln\left(x+\sqrt{1+x^2}\right)$$

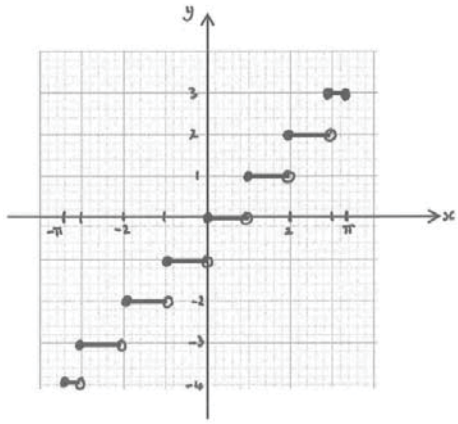
and hence $\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x = \left(\frac{1}{2}x^2+\frac{1}{4}\right)\ln\left(x+\sqrt{1+x^2}\right) - \frac{1}{4}x\sqrt{1+x^2}$ (+ C).

Question 3

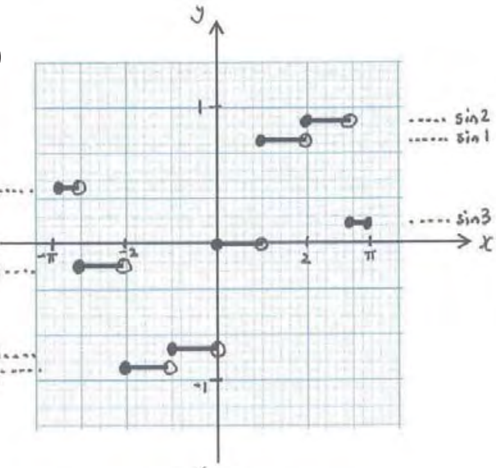
If you have not seen this sort of function before, then it is worthwhile playing around with such things as part of your preparation for the STEPs, which frequently test perfectly simple ideas in contexts that are not a standard part of A-level (or equivalent) courses. Being able to think things through calmly and carefully under examination conditions is an especially high-level skill, but one that can be practised.

In this case, the "integer part" function is a relatively simple one to deal with, as it only changes values when the function it acts on hits an integer value. Before commencing work on this question, note that the "integer part" of a negative number is the one to the left of it (if it lies between integers, of course), and many function-plotting packages are set to "go right" for negative numbers, which is unfortunate. There is also the small matter of how to illustrate the "*y*" values at those points when the "step" occurs … the tradition is to employ a "filled" circle (●) for inclusion and an "open" circle (○) for exclusion. With a bit of care you should find that the four graphs required here look as follows.

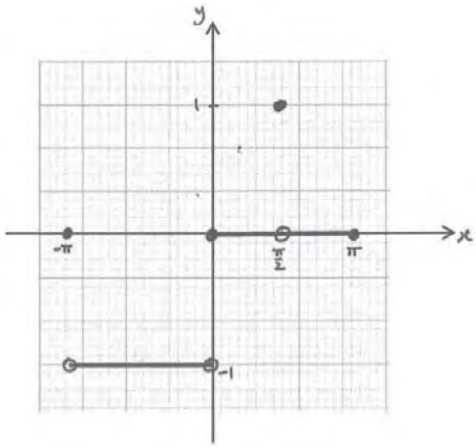
(i)



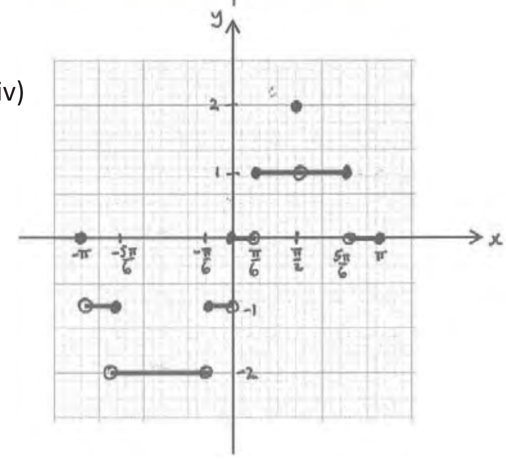
(ii)



(iii)



(iv)



Question 4

As with Q2, this begins with a simple instruction to differentiate; again, the idea is that you will tidy up the final answer for later reference. Using the *Quotient* and *Chain Rules* on $y = \dfrac{z}{\sqrt{1+z^2}}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\sqrt{1+z^2}.1 - z.\tfrac{1}{2}\left(1+z^2\right)^{-\frac{1}{2}}.2z}{\left(\sqrt{1+z^2}\right)^2}$$

which simplifies to $\dfrac{1}{\left(1+z^2\right)^{\frac{3}{2}}}$.

The given expression in part (ii) initially appears to be quite awful, until you realise that writing, for instance, $z = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ turns $\dfrac{\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)}{\left\{1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{\frac{3}{2}}} = \kappa$ into $\dfrac{\dfrac{\mathrm{d}z}{\mathrm{d}x}}{\left(1+z^2\right)^{\frac{3}{2}}} = \kappa$, and this can now be seen to be a standard "separable variables" first-order differential equation: $\displaystyle\int \frac{\mathrm{d}z}{\left(1+z^2\right)^{\frac{3}{2}}} = \int \kappa\,\mathrm{d}x$. Using (i)'s result then gives $\dfrac{z}{\sqrt{1+z^2}} = \kappa(x+c)$ (where the usual "+ c" has been incorporated into a slightly more helpful form here).

Re-arranging this for z or z^2 leads to $z^2 = \kappa^2(x+c)^2(z^2+1) \Rightarrow z = \pm\dfrac{u}{\sqrt{1-u^2}}$ where (again) the more complicated-looking term has been given a new label, which is a simple but effective device to make what to do next more obvious: here, $u = \kappa(x + c)$.

We now substitute back for $z = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}.\dfrac{\mathrm{d}u}{\mathrm{d}x}$ and use the *Chain Rule* (e.g.) with $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \kappa$ to obtain another "separable variables" first-order differential equation, $\kappa\dfrac{\mathrm{d}y}{\mathrm{d}u} = \pm\dfrac{u}{\sqrt{1-u^2}}$ or $\displaystyle\int \kappa\,\mathrm{d}y = \pm\int \frac{u}{\sqrt{1-u^2}}\mathrm{d}u$. At this point, you should be able to see that $\displaystyle\int \frac{u}{\sqrt{1-u^2}}\mathrm{d}u$ can be integrated (by "recognition", "reverse chain rule" or a substitution) to give $-\sqrt{1-u^2}$. Substituting for u then gives $ky + d = \mp\sqrt{1-\kappa^2(x+c)^2}$ and squaring both sides leads towards a circle equation $(\kappa y + d)^2 = 1 - \kappa^2(x+c)^2$ or $(x+c)^2 + \left(y + \dfrac{d}{\kappa}\right)^2 = \left(\dfrac{1}{\kappa}\right)^2$, which is the equation of a circle, with centre $\left(-c, -\dfrac{d}{\kappa}\right)$ and a radius which is the reciprocal of the curvature .

Question 5

This sort of situation is relatively common in '*Maths Challenges*' and the usual approach is to join all the circles' centres to the points of tangential contact and then form some right-angled triangles by considering (here) the horizontal line through B's centre. Because of the well-known (GCSE) *Circle Theorem* "tangent perpendicular to radius" result, it is the case that each of *AB*, *BC* and *CA* is a straight line. This enables us to use *Pythagoras' Theorem*:

$$PR = PQ + QR \Rightarrow \sqrt{(a+c)^2-(a-c)^2} = \sqrt{(b+a)^2-(b-a)^2} + \sqrt{(c+b)^2-(c-b)^2}$$

which simplifies to $\sqrt{4ac} = \sqrt{4ab} + \sqrt{4bc}$ and, upon division throughout by $\sqrt{4abc}$, gives the required answer $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}$.

There are many ways to approach the next result, but it should be clear that each will, at some stage, require the replacement of the *b*'s with *a*'s and *c*'s (or equivalent). The most direct route would be to examine the LHS and RHS of (**) separately, and then show that they match up. This would look like:

LHS = $2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \frac{2}{a^2}+\frac{2}{c^2}+2\left(\frac{1}{a^2}+\frac{4}{a\sqrt{ac}}+\frac{6}{ac}+\frac{4}{c\sqrt{ac}}+\frac{1}{c^2}\right) = \frac{4}{a^2}+\frac{12}{ac}+\frac{4}{c^2}+\frac{8}{a\sqrt{ac}}+\frac{8}{c\sqrt{ac}}$.

RHS = $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 = \left(\frac{1}{a}+\left\{\frac{1}{a}+\frac{2}{\sqrt{ac}}+\frac{1}{c}\right\}+\frac{1}{c}\right)^2 = \left(\frac{2}{a}+\frac{2}{\sqrt{ac}}+\frac{2}{c}\right)^2 = 4\left(\frac{1}{a}+\frac{1}{\sqrt{ac}}+\frac{1}{c}\right)^2$

$= 4\left(\frac{1}{a^2}+\frac{3}{ac}+\frac{1}{c^2}+\frac{2}{a\sqrt{ac}}+\frac{2}{c\sqrt{ac}}\right)$, and these are clearly the same.

Working in the other direction is trickier, but not much more so, and it is again helpful to re-label the variables to make things *look* simpler, especially if we can somehow remove the need for everything to appear as a fraction. So, following an initial observation that

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}$$

we could write $x = \frac{1}{\sqrt{a}}$, $y = \frac{1}{b}$, $z = \frac{1}{\sqrt{c}}$, so that we are now trying to prove that

$$x^4 + y^4 + z^4 = 2x^2y^2 + 2y^2z^2 + 2z^2x^2.$$

(Although it is not essential to do this at this stage, it is often the case that folks forget to do it at the end if they don't; and that is to consider the given conditions $b < c < a$, which translate to $y > z > x$.)

Now, completing the square: $\left(x^2+z^2-y^2\right)^2 = 4x^2z^2 \Leftrightarrow x^2+z^2-y^2 = \pm 2xz \Leftrightarrow (z \mp x)^2 = y^2$, and there are the four cases to consider: $y = x - z$, $y = z - x$, $y = x + z$ or $y = -x - z$. Consideration of the above conditions on *x*, *y*, *z* then shows that only $y = x + z$ is suitable, and so $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{a}}$, as required.

Question 6

This is a fairly straightforward vectors question that involves little beyond working with the vector equation of a line. To begin with, you are required to explain a couple of introductory results that rely on the fact that two (non-zero) vectors are multiples of each other if and only if they are parallel. Thus, $OX \,||\, OA \Rightarrow \mathbf{x} = m\mathbf{a}$, with $0 < m < 1$, since X is between O and A; and $BC \,||\, OA \Rightarrow \mathbf{c} - \mathbf{b} = k\mathbf{a}$ and so $\mathbf{c} = k\mathbf{a} + \mathbf{b}$, with $k < 0$ since BC is in the opposite direction to OA.

Then, lines OB and AC have vector equations $\mathbf{r} = \beta\,\mathbf{b}$ and $\mathbf{r} = \mathbf{a} + \alpha\,(\mathbf{c} - \mathbf{a})$ respectively, for some scalar parameters α and β. Replacing $\mathbf{c}$ by $k\mathbf{a} + \mathbf{b}$ and equating the two $\mathbf{r}$'s for the point of intersection then gives $\beta\,\mathbf{b} = \mathbf{a} + \alpha\,(k\mathbf{a} + \mathbf{b} - \mathbf{a})$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, we can equate terms to find that $1 - \alpha + \alpha k = 0$ and $\alpha = \beta$. Solving leads to $\alpha = \beta = \dfrac{1}{1-k}$, so that $\mathbf{d} = \dfrac{1}{1-k}\mathbf{b}$.

In an exactly similar way, we then have $Y = XD \cap BC \Rightarrow m\mathbf{a} + \alpha\left(\dfrac{1}{1-k}\mathbf{b} - m\mathbf{a}\right) = \mathbf{b} + \beta k\,\mathbf{a}$. (Note that there is no reason why we have to use different symbols for the scalar parameters each time, as they are of no actual significance in relation to the problem.) Equating coefficients $\Rightarrow m - \alpha m - \beta k = 0$ and $\dfrac{\alpha}{1-k} = 1$, so that $\mathbf{y} = km\mathbf{a} + \mathbf{b}$.

Next, $Z = OY \cap AB \Rightarrow (1 - \alpha)\mathbf{a} + \alpha\mathbf{b} = \beta\left(km\mathbf{a} + \mathbf{b}\right)$, and so $1 - \alpha - km\beta = 0$ and $\alpha = \beta \left(= \dfrac{1}{1+km}\right)$, giving $\mathbf{z} = \left(\dfrac{km}{1+km}\right)\mathbf{a} + \left(\dfrac{1}{1+km}\right)\mathbf{b}$; and $T = DZ \cap OA \Rightarrow \alpha\,\mathbf{a} = \dfrac{1}{1-k}\mathbf{b} + \beta\left(\dfrac{km}{1+km}\mathbf{a} + \dfrac{1}{1+km}\mathbf{b} - \dfrac{1}{1-k}\mathbf{b}\right)$, whence $\alpha = \dfrac{\beta km}{1+km}$ and $0 = \dfrac{1-\beta}{1-k} + \dfrac{\beta}{1+km}$, so that $\mathbf{t} = \left(\dfrac{m}{1+m}\right)\mathbf{a}$.

Notice that all that has been done so far is to work out the position vectors of the points created as the intersections of various pairs of lines. If this is difficult to visualize, then a diagram should be drawn first.

All that remains is to set up the lengths of the various line segments of interest. If we call $OA = a$, then it follows that $OX = ma$, $OT = \left(\dfrac{m}{1+m}\right)a$, $TX = \left(\dfrac{m^2}{1+m}\right)a$, $TA = \left(\dfrac{1}{1+m}\right)a$ and $XA = (1 - m)a$. (Note that the shrewd solver would simply take, w.l.o.g., the value of a to be 1, as it is an arbitrary length and cancels throughout any of the working that follows in order to obtain the two given answers,

$$\frac{1}{OT} = \frac{1}{a}\left(1 + \frac{1}{m}\right) = \frac{1}{OA} + \frac{1}{OX} \text{ and } OT.OA = \left(\frac{m}{1+m}\right)a^2 = (ma).\left(\frac{1}{1+m}\right)a = OX.TA.$$

Question 7

Firstly, $S \cup T$ = the set of all positive odd numbers; and $S \cap T = \phi$.

Next, we must show that the product of two different elements of *S* is also an element of *S*, and then do a similar thing for two different elements of *T*. This involves noting that

$$(4a + 1)(4b + 1) = 4(4ab + a + b) + 1,$$

which is an element of *S*; and

$$(4a + 3)(4b + 3) = 4(4ab + 3a + 3b + 2) + 1,$$

which is, in fact, in *S* rather than *T*.

The result of part (iii) essentially requires proof by contradiction, so we first suppose that *t* is an element of *T*, and that all of *t*'s prime factors are in *S*. Noting that there are no even factors, we can write

$$t = (4a + 1)\,(4b + 1)\,(4c + 1)\,\ldots\,(4n + 1).$$

But we have already noted that the product of any pair of elements of *S* will always yield another element of *S*, and hence (inductively, I suppose), $t = 4\{\ldots\ \ldots\ \ldots\} + 1$ is always in *S*. Which contradicts the assumption that *t* is an element of *T.*

For part (iv) (a), we note that an element of *T* is either *T*-prime or *T*-composite. If it is the latter, then it can be expressed as a product of *T*-primes. However, we have already established that every pair of factors in *T* multiply to give an element of *S*, as do every pair of elements of *T*. So, after every pairing-up of this element's factors, there must be an odd one left over to multiply by in order to give an element of *T*. Hence, altogether, there is an odd number of them.

For the final part of the question, we are required to find non-primes in *S* that are products of elements of *T* that can be "re-grouped" suitably. One such example involves the numbers 9, 21, 33 and 77, each of which is both *S*-prime and a product of elements in *T*:

$$9 = 3 \times 3,\ 21 = 3 \times 7,\ 33 = 3 \times 11 \text{ and } 77 = 7 \times 11; \text{ with 3, 7 and 11 in } T.$$

This leads to the example $9 \times 77 = 21 \times 33$ (= 693). Of course, the main purpose of the question is to demonstrate the existence of perfectly reasonable number-sets (in this case, *S*), having the standard properties of multiplication, yet for which the "unique factorisation" principle no longer holds. This is a very important principle relating to prime numbers within the set of positive integers, which you have been taught (quite rightly) to assume, but that I imagine you have never had reason to think that it might not necessarily hold in this case, or in other similar situations.

Question 8

This is a simple idea involving terms of series. The most important thing here is to be sure that you can justify the form of the n^{th} term for any given series. To begin with, it is helpful to realise that the *Binomial Theorem* applied to the negative integers, gives the following results:

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$, with the coefficient of x^n being 1;

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$, with the coefficient of x^n being $(n + 1)$; and

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \tfrac{1}{2}(n+1)(n+2)x^n + \ldots$, having the *triangular numbers* as coefficients; etc.

Having established these results, it is easy to show that

$$0 + x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots = x(1-x)^{-2}$$

and that

$$x(1-x)^{-3} = x\left(1 + 3x + 6x^2 + 10x^3 + \ldots + \tfrac{1}{2}n(n+1)x^{n-1} + \ldots\right) = 0 + x + 3x^2 + 6x^3 + \ldots + \tfrac{1}{2}n(n+1)x^n + \ldots$$

has coefficient of x^n equal to $u_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n$.

Using these first two results: $2 \times (2^{\text{nd}}) - (1^{\text{st}})$ gives $\dfrac{2x}{(1-x)^3} - \dfrac{x}{(1-x)^2}$ with $u_n = n^2$.

There are several ways to proceed with part (ii) (a); the simplest being to note that

$$a + akx + ak^2x^2 + ak^3x^3 + \ldots + ak^nx^n + \ldots = a + kx\left(a + akx + ak^2x^2 + ak^3x^3 + \ldots + ak^nx^n + \ldots\right)$$

$$= a + kx\,\mathrm{f}(x)$$

and so $\mathrm{f}(x) = a\left(\dfrac{1}{1-kx}\right)$.

For part (ii) (b), you should note that the given *second*-order recurrence relation (i.e. *two* preceding terms are involved) requires two starting terms before things get going systematically, so it is best to split off the first two terms of the series before attempting to make use of this defining feature.

Writing $\mathrm{f}(x) = 0 + x + \sum_{n=2}^{\infty} u_n x^n = 0 + x + \sum_{n=2}^{\infty} (u_{n-1} + u_{n-2})x^n = x + x\sum_{n=2}^{\infty} u_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} u_{n-2}x^{n-2}$

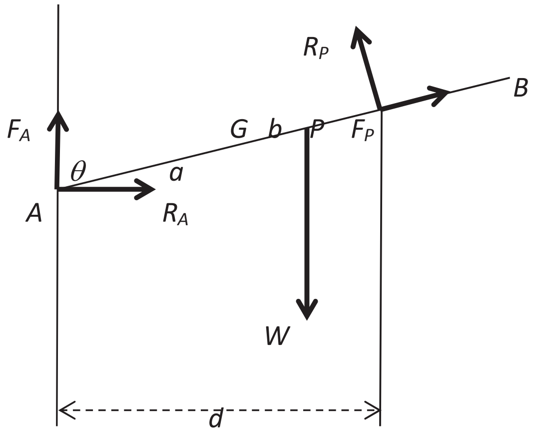
Note: we are trying to re-create f(x) on the right-hand side

$$= x + x\sum_{n=1}^{\infty} u_n x^n + x^2\sum_{n=0}^{\infty} u_n x^n = x + x\{\mathrm{f}(x) - 0\} + x^2\mathrm{f}(x)$$

so that $\mathrm{f}(x) = \dfrac{x}{1 - x - x^2}$.

Question 9

As with all *Statics* problems, it is so important to have a clear diagram, suitably marked with all the relevant forces (and preferably in the correct directions). The given diagram appears to give the situation when the horizontal rail is more than half-way along the rod; in this configuration, the rod will slip so that the end *A* slides down the wall, and to the left relative to the contact at *P* (as shown below).



The next essential part of any solution is to write down all the key statements before attempting to process them in some way. The guidance "resolve twice and take moments" is surely a stock part of every mechanics teacher's repertoire, and it is sound advice. It is customary to resolve in two perpendicular directions (vertically and horizontally here) and to find some point about which to take moments that minimises the "clutter" of subsequent algebra-and-trigonometry (*A* has been chosen here). The additional use of the *Friction Law* for, on this occasion, the case of limiting equilibrium at the two points of contact is also needed. This gives us the following "ingredients" for use in a solution.

Friction Law $F_A = \lambda R_A$ and $F_P = \mu R_P$.

Res.↑ $W = F_A + R_P \sin\theta + F_P \cos\theta$

Res.→ $R_A = R_P \cos\theta - F_P \sin\theta$

↻*A* $W a \sin\theta = R_P (a + b)$

Eliminating the *F*'s from the two resolving statements gives

$$W = \lambda R_A + R_P \sin\theta + \mu R_P \cos\theta \quad \text{and} \quad R_A = R_P \cos\theta - \mu R_P \sin\theta$$

and introducing *d* in the moments statement (noting that it needs to appear in the answer) gives

$$W a \sin^2\theta = R_P d.$$

Since this last equation involves just the forces *W* and R_P it makes sense to eliminate R_A next, to get

$$W = R_P(\lambda\cos\theta - \lambda\mu\sin\theta + \sin\theta + \mu\cos\theta).$$

Finally, dividing these last two leads to the given answer,

$$d\operatorname{cosec}^2\theta = a([\lambda+\mu]\cos\theta + [1-\lambda\mu]\sin\theta).$$

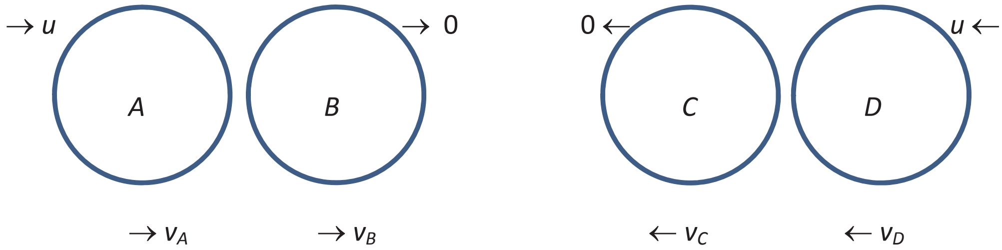
For the case when *P* is less than half-way along the rod, which will now slide up the wall at *A* (etc.), we can simply write $F_A \to -F_A$; $F_P \to -F_P$; $a + b \to a - b$, or just switch the signs of λ and μ, as everything else remains unchanged. The corresponding result is thus shown to be

$$d\operatorname{cosec}^2\theta = a(-[\lambda+\mu]\cos\theta + [1-\lambda\mu]\sin\theta).$$

Question 10

When it comes to it, collisions questions involve only the use of two (sometimes three, if energy considerations are involved) principles. In this case, the principles of *Conservation of Linear Momentum* (CLM) and *Newton's (Experimental) Law of Restitution* (NEL or NLR). Diagrams are also quite important in these types of questions, but largely to enable the solver to be clear about what directions are being taken as positive: remember that velocity and momentum are vectorial in nature. If you are ever unsure about which direction any of the objects (particles or spheres, etc.) will be moving in after the collision has taken place, then just have them all going the same way; this makes it much easier to interpret negative signs in any later answers.

To begin with, there are two separate collisions, between A/B and C/D, as shown below.



For collision A/B **CLM**: $m(\lambda u = \lambda v_A + v_B)$ and **NEL**: $eu = v_B - v_A$

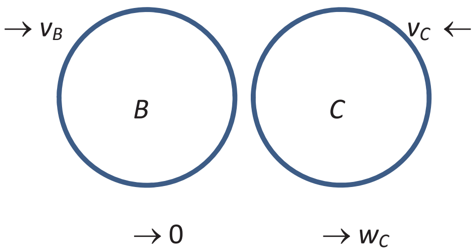
and for collision C/D **CLM**: $m(u = v_C + v_D)$ and **NEL**: $eu = v_C - v_D$

Solving each pair of equations, separately, gives

$$v_A = \frac{\lambda - e}{\lambda + 1}u \text{ , } v_B = \frac{\lambda(1+e)}{\lambda+1}u \text{ , } v_C = \tfrac{1}{2}(1+e)u \text{ and } v_D = \tfrac{1}{2}(1-e)u \text{ ;}$$

even though some of these turn out not to be required (though they may be later on, of course).

There is then a further collision between B/C:



CLM: $m(v_B - v_C) = m\, w_C$ and **NEL**: $e(v_B + v_C) = w_C$

Now, substituting previous answers in terms of e and u, and identifying for e, leads to the required answer $e = \dfrac{\lambda - 1}{3\lambda + 1}$. To justify the following condition on e, note that

$$e = \frac{\lambda - 1}{3\lambda + 1} = \tfrac{1}{3}\left(\frac{3\lambda - 3}{3\lambda + 1}\right) = \tfrac{1}{3}\left(\frac{3\lambda + 1 - 4}{3\lambda + 1}\right) = \tfrac{1}{3} - \frac{\tfrac{4}{3}}{3\lambda + 1} < \tfrac{1}{3}$$

since the term being subtracted is positive. (It is not sufficient simply to show that $e \to \tfrac{1}{3}$.)

Finally, using $w_C = \frac{(1+e)(\lambda-1)}{2(\lambda+1)}u$ from previous work, and equating this to v_D, gives

$$\frac{(1+e)(\lambda-1)}{2(\lambda+1)}u = \tfrac{1}{2}(1-e)u\,;$$

and substituting for e (e.g.) enables us to solve for λ and then find e: $\lambda = \sqrt{5}+2$, $e = \sqrt{5}-2$.

Question 11

Projectiles questions are often straightforward in principle, but can often require both careful thought and some tricky trigonometric manipulation. Occasionally, there is the additional requirement to maximise or minimise something-or-other using calculus.

To begin with, it is useful to know (or to be able to derive quickly) the *Trajectory Equation* of the parabola: $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$. Setting $y = -h$ and re-arranging then gives $\dfrac{gx^2}{u^2} = 2h\cos^2\alpha + 2x\sin\alpha\cos\alpha$, and it should be fairly clear from the given answer that a little bit of work using the double-angle formulae will lead to the required result $\dfrac{gx^2}{u^2} = h(1+\cos 2\alpha) + x\sin 2\alpha$.

Differentiating w.r.t. $\alpha \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}\alpha}\left(\dfrac{gx^2}{u^2}\right) = h(-2\sin 2\alpha) + \left(x.2\cos 2\alpha + \sin 2\alpha.\dfrac{\mathrm{d}x}{\mathrm{d}\alpha}\right)$. Notice that there is actually no need to re-arrange for $\dfrac{\mathrm{d}}{\mathrm{d}\alpha}(\ldots)$ since we require both derivative terms to disappear. This immediately leads to $x = h\tan 2\alpha$; and, substituting back, we get $\dfrac{gh^2\tan^2 2\alpha}{u^2} = h(1+\cos 2\alpha) + h\tan 2\alpha\sin 2\alpha$. By cancelling one of the h's and (e.g.) writing all trig. terms in $c = \cos 2\alpha$ then yields

$$\frac{gh(1-c^2)}{u^2c^2} = 1 + c + \frac{1-c^2}{c} \Rightarrow gh - ghc^2 = u^2(c^2 + c^3 + c - c^3).$$

As a quadratic in c : $0 = (u^2+gh)c^2 + u^2c - gh$ = $[(u^2+gh)c - gh](c+1)$ … note that it is always worth trying to factorise before deploying the messier *quadratic formula*.

We now have $\cos 2\alpha = \dfrac{gh}{u^2+gh}$, and substituting $x = h\tan 2\alpha$ and $y = -h$ in $\Delta^2 = x^2 + y^2$, then gives

$$\Delta^2 = h^2\sec^2 2\alpha \quad \text{i.e. } \Delta = h\sec 2\alpha$$

so that $\Delta = h.\dfrac{u^2+gh}{gh} = \dfrac{u^2}{g} + h$, as required.

Question 12

Bizarrely, this probability question concerning tossing fair coins has all of its answers equal to $\frac{1}{2}$. As one might imagine, such an answer can be obtained in very many ways indeed and so the key is to make your working really clear as to how the answer is arrived at. Merely writing down a whole load of fractions on the page is really not very enlightening. There needs to be some (visible) systematic approach to counting cases, followed by the numerical work that goes with it. For instance, in part (i), we could break down the answer into each possible value for A and the values of B that could then go with it.

e.g. p(A = 0).p(B = 1,2 or 3) + p(A = 1).p(B = 2 or3) + p(A = 2).p(B = 3) = $\frac{1}{4}\times\frac{7}{8}+2\times\frac{1}{4}\times\frac{4}{8}+\frac{1}{4}\times\frac{1}{8}$ = $\frac{1}{2}$.

For part (ii), one should by now have been able to "see" how the results arise, and can appeal to a "similar" process; e.g. $\frac{1}{8}\times\left(\frac{4+6+4+1}{16}\right)+\frac{3}{8}\times\left(\frac{6+4+1}{16}\right)+\frac{3}{8}\times\left(\frac{4+1}{16}\right)+\frac{1}{8}\times\left(\frac{1}{16}\right)$ … notice the appearance of the binomial coefficients for counting the relevant numbers of B's possible values. This gives $\frac{1}{128}\left(15+3\times11+3\times5+1\right)=\frac{1}{128}\left(64\right)$ = $\frac{1}{2}$.

In part (iii), you should note that, when each of them has tossed n coins,

p(B has more Hs) = p(A has more Hs) = p_2

and that p(A_H = B_H) = p_1 . Thus $p_1 + 2p_2 = 1$.

Next, considering what happens when B tosses the extra coin,

p(B has more Hs) = p(B already has more Hs) × p(B gets T)

+ p(B already has more, or equal, Hs) × p(B gets H)

= $p_2\times\frac{1}{2}+(p_1+p_2)\times\frac{1}{2}$ = $\frac{1}{2}(p_1+2p_2)=\frac{1}{2}$.

Question 13

For the i-th e-mail, the pdf is $f_i(t) = \lambda e^{-\lambda t}$. Integrating this gives the cdf: $F_i(t) = -e^{-\lambda t} + C$, and $F(0) = 0 \Rightarrow C = 1$.

Then, for n e-mails sent simultaneously,

$F(t) = P(T \le t) = 1 - P(\text{all } n \text{ take longer than } t) = 1 - \left(e^{-\lambda t}\right)^n$,

(using the product of n independent probabilities)

$= 1 - \lambda e^{-\lambda n t}$.

Differentiating this then gives the required pdf of T, $f(t) = n\lambda e^{-\lambda n t}$.

Finding an expected value is a standard integration process: $E(T) = \int_0^\infty t \times n\lambda e^{-\lambda n t}\, dt$, and this requires the use of integration by parts: $E(T) = \left[-te^{-\lambda n t}\right]_0^\infty + \int_0^\infty n\lambda e^{-\lambda n t}\, dt = [0] + \left[\dfrac{-e^{-\lambda n t}}{\lambda n}\right]_0^\infty = \dfrac{1}{n\lambda}$.

For the very final part, one *could* go again through the route of pdfs and cdfs, but it should be obvious that the waiting-time for the 2nd email is simply the 1st from the remaining $(n - 1)$ … with expected arrival time $\dfrac{1}{(n-1)\lambda}$, giving a total waiting time of $\dfrac{1}{n\lambda} + \dfrac{1}{(n-1)\lambda} = \dfrac{1}{\lambda}\left(\dfrac{1}{n} + \dfrac{1}{(n-1)}\right)$.

STEP II 2016 Solutions

Question 1

Use t_1 and t_2 to represent the value of the parameter t at each of the points P and Q. The equations of the two tangents can therefore be found in terms of t_1 and t_2 and the fact that POQ is a right angle can be used to find a relationship between t_1 and t_2. The point of intersection of the two tangents can therefore be found in terms of just t_1 and this is a pair of parametric equations for the curve that the point of intersection makes.

Substituting the parametric equations for C_1 into the equation for C_2 gives a cubic equation in t^2 which can be solved by inspection to show that there are just two intersections and so the two curves just touch, but do not cross.

Question 2

Substitute $c = a + b$ into the expression to show that $a + b - c$ is a factor. Once this is done, the symmetry shows that $b + c - a$ and $c + a - b$ must also be factors and therefore there is just a constant multiplier that needs to be deduced to obtain the full factorisation of (*).

For part (i), choices of a, b and c need to be made so that

$$a + b + c = x + 1$$

$$a^2 + b^2 + c^2 = \frac{2x^2 + 5}{2}$$

$$a^3 + b^3 + c^3 = \frac{4x^3 + 13}{4}$$

Once these have been identified the solutions to the equation follow from the factorisation already deduced.

Once the substitution $d + e = c$ has been made it is only necessary to identify the parts of the expression which differ from (*) in the first part of the question (which arise from the c^2 and c^3 terms). The factorisation and solution of the equation then follow a similar process to the first part of the question.

Question 3

The differentiation to show the result in part (i) should not present much difficulty, although it is important to show that all of the terms (and no others) are present.

For part (ii) observe that each individual term of $f_n(x)$ has a positive coefficient, so for any positive value of x the value of $f_n(x)$ must be positive.

For part (iii), use the result in part (i) to rewrite $f_n{}'(x)$ in terms of $f_n(x)$ and note that $f_n(a)$ and $f_n(b)$ must be 0. This means that any pair of roots must have a gradient of the same sign, which leads to an argument that there must be another root between the two. As this would lead to an infinite number of roots to a polynomial, there cannot be more than one root.

To establish the number of roots in the two cases consider the behaviour of the graph as $x \to \infty$ and as $x \to -\infty$

Question 4

The equation given can be rewritten as a quadratic in x. The discriminant then establishes the required result. To show the second result, show that $y^2 + 1 \geq (y\cos\theta - \sin\theta)^2$, which can be shown by writing $y\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ and then this result is a quadratic inequality that leads directly to the next result.

In the case $y = \frac{4+\sqrt{7}}{3}$, careful manipulation of surds shows the required result and so the value of θ must be the value of α obtained in the previous section. Finally, the value of x can be obtained by returning to the original equation and substituting in the values that are known.

Question 5

The binomial expansion for $(1-x)^{-N}$ should be easy enough, it is then required to write the product in terms of factorials so that the expression can be written in terms of $\binom{p}{q}$.

Since the expansion of $(1-x)^{-1}$ involves a coefficient of 1 for every term, the coefficient of x^n in the expansion of $(1-x)^{-1}(1-x)^{-N}$ is simply the sum of the coefficients of all of the terms in the expansion of $(1-x)^{-N}$ up to and including the term in x^n.

The products in the sum on the right-hand side of the result in part (ii) should be recognisable as binomial coefficients in the case where the power is a positive integer, so use

$$(1+x)^p(1+x)^q \equiv (1+x)^{p+q}$$

and compare coefficients as in part (i).

Similarly for part (iii), identify that the result will come from consideration of

$$(1+x)^{N+m}(1+x)^{-m} \equiv (1+x)^N.$$

Question 6

Parts (i) and (ii) only require verification in each of the cases, so simply differentiate the functions given and substitute into the differential equation to confirm that they are solutions. Remember to check as well that the boundary conditions are satisfied.

For part (iii), differentiate the given formula for z and substitute into the differential equation. By observing that the new differential equation is of the same form as (*), but for $2n$ instead of n, the expression for $y_{2n}(x)$ can be established.

For part (iv), again differentiate the given formula, being careful about the application of the chain rule and substitute. Again, by comparing with (*) the final result should be clear.

Question 7

The first result can be shown by using a substitution into the integral, being careful to explain the change of sign when the limits of the integral are switched.

Simple application of knowledge of trigonometric graphs once the substitution has been made can be used to show that twice the integral is equivalent to integrating the function 1 over the interval.

Similarly, the remaining integrals can all be rearranged using standard trigonometric identities and knowledge of logarithms into forms that can be integrated from standard results once the substitution from (*) has been made.

Question 8

The integral required at the start of the question should be a straightforward one to evaluate. When making a sketch to illustrate the result in the second part, ensure that the sum is indicated by a series of rectangles, with the graph of the curve passing through the midpoints of the tops.

In part (i), the integral that would match the sum given results in an answer of 2, so this is the first of the estimates. The remaining estimates arise from using the integral to estimate most of the sum, but taking the first few terms as the exact values (so in each case the integration is taken from a different lower limit).

For part (ii), evaluate the integral for one particular term of the sum and note that it is approximately $\frac{1}{4r^4}$. Finally, using the most accurate estimate for $E\left(\frac{33}{20}\right)$ the sum from $r = 3$ onwards can be calculated and then the first two values of $\frac{1}{r^4}$ can be added to achieve the desired result.

Question 9

The result in part (i) follows from consideration of kinetic energy lost and work done.

In part (ii) apply conservation of momentum to the combined block and bullet after the bullet hits the block. By comparing to the case in part (i) the motion of the bullet until it is at rest relative to the block can be analysed. Once all of the relevant equations of motion have been written down, a series of simultaneous equations will have been found from which the values of b and c can be found.

Question 10

The first requirement will be to find the centre of mass of the triangle. Once this is done a diagram will be very useful and notations will need to be added for various distances, angles and the frictional force. From this diagram the forces can be resolved in two perpendicular directions and moments can be taken. This leads to a series of equations which can be solved to work out the value that the frictional force would have to take to prevent slipping. From this the required result can be established.

Question 11

The particles must collide if they would be in the same position for one particular value of t. Therefore, writing out the equations of motion for the two particles and eliminating the variables that are not needed the required result can be reached.

For the second part, the time of the collision can be found by considering the heights of the bullet and target at time t and noting that these must be equal. Once the value of t has been found, the fact that this must be positive leads to the inequality that is required for the first result.

For the final part, note that gravity affects both the bullet and target in the same way, so if it is ignored then the time of collision (if there is one) will be the same and this is a situation as in part (i). Clearly, in part (i) the two objects must be moving towards each other if there is to be a collision.

Question 12

Replace B with $(B \cup C)$ in the result that you must start with and then observe that $A \cap (B \cup C)$ is the same as $(A \cap B) \cup (A \cap C)$. The corresponding result for four events should be clear, but care must be taken to include all of the possible pairs.

The results for parts (i), (ii) and (iii) should be clear from consideration of arrangements in each case and the result required follows from the generalisation of the result from the start of the question.

The probability that the first card is in the correct position and none of the others is can be established and therefore the probability that exactly one card is in the correct position will be n times that.

Question 13

For part (i) the approximation of the binomial distribution by a normal distribution should be known and the area under the curve (applying a continuity correction) can then be approximated by a rectangle.

The second result follows from a similar approximation and the use of the formula for a probability from the binomial distribution.

Part (iii) follows from an approximation of a Poisson distribution with a normal distribution and again approximating the required area by a rectangle.

STEP III 2016 Solutions

1. Part (i) is most simply dealt with by the suggested method, change of variable, and it is worth completing the square in the denominator to simplify the algebra leading to a trivial integral. Part (ii) can either be attempted immediately using integration by parts, starting from I_n and obtaining $\int_{-\infty}^{\infty} \frac{2nx(x+a)}{(x^2+2ax+b)^{n+1}}\, dx$ and then writing the numerator as $2n(x^2+2ax+b) - na(2x+2a) - 2n(b-a^2)$. Alternatively, use of the same substitution in I_{n+1} as in part (i) leads to the need to integrate $\cos^{2n} u$, which in turn can be written as $\cos^{2n-2} u(1-\sin^2 u) = \cos^{2n-2} u - \cos^{2n-2} u \sin u. \sin u$, with the second term being susceptible to integration by parts. Part (iii) follows from the previous parts by induction using part (ii) to achieve the inductive step and (i) the base case.

2. There are numerous correct ways through this question. Working parametrically with $x = at^2$, $y = 2at$ gives a normal as $tx + y = at^3 + 2at$ and imposing that this passes through $(ap^2, 2ap)$ yields $t^2 + tp + 2 = 0$ (*) which has roots q and r, the former giving (i). As a consequence, $r + q = -p$ and $rq = 2$, so that QR, $2x - (r+q)y + 2aqr = 0$, simplifies to $2x + py + 4a = 0$, and thus passes through $(-2a, 0)$ for (ii). T can be shown to be $\left(-a, \frac{-2a}{p}\right)$, which, of course, lies on $x = -a$, and as (*) had two real distinct roots, q and r, $p^2 - 8 > 0$, which yields $\left|\frac{-2a}{p}\right| < \frac{a}{\sqrt{2}}$.

3. Differentiating, multiplying by denominators and dividing by the exponential function, gives $[Q(P+P') - PQ'](x+1)^2 = (x^3-2)Q^2$ which, invoking the factor theorem, gives the first required result. Denoting the degree of P by p and that of Q by q in this expression yields $p + q + 2 = 2q + 3$ and hence the desired result in (i). Furthermore, in the given case, substitution in the same result and postulating $P(x) = ax^2 + bx + c$ yields consistent equations for a, b and c and thus $P(x) = x^2 - 2x$.
For part (ii), commencing as in part (i) demonstrates again that Q has a factor $(x+1)$ as $[Q(P+P') - PQ'](x+1) = Q^2$. Supposing $Q(x) = (x+1)^n S(x)$, where $n \geq 2$ and $S(-1) \neq 0$, with $P(-1) \neq 0$ and substituting in the expression already derived leads to a contradiction.

4. The considered expression equates to $\frac{(x-1)x^r}{(1+x^r)(1+x^{r+1})}$ and so, by the method of differences,

$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{1}{(x-1)}\left[\frac{1}{1+x} - \frac{1}{1+x^{N+1}}\right]$, and letting $N \to \infty$, the desired result is obtained. Writing $\operatorname{sech}(ry)$ as $\frac{2e^{-ry}}{1+e^{-2ry}}$ and similarly $\operatorname{sech}\big((r+1)y\big)$, the result of part (i) with $x = e^{-2y}$ can be used to obtain the result. Care needs to be taken to write $\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big)$ as

$2\left[\sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}\big((r+1)y\big) + \operatorname{sech} y\right]$ which with the previous deduction of (ii) can be simplified to $2 \operatorname{cosech} y$.

5. The binomial expansion, evaluated for $x = 1$, appreciating that terms are symmetrical contains two terms equal to the LHS of the inequality, and so truncating to them gives double the required result in (i). Appreciating that $\frac{(2m+1)!}{(m+1)!m!}$ is an integer and that if $m + 1 < p \leq 2m + 1$, with p a prime, implies p divides the numerator and not the denominator of this expression and hence divides the integer then can be extended for all such primes yielding the result, with the deduction following from (i). For (iii), it can be shown that $m + 1 \leq 2m$ and writing $P_{1,2m+1}$ as

$P_{1,m+1}P_{m+1,2m+1}$, combining the given result and (ii), the desired result is obtained. Part (iv) is obtained by use of strong induction with the supposition, $P_{1,m} < 4^m$ for all $m \leq k$ for some particular $k \geq 2$, and considering the cases k even and odd separately and making use of (iii).

6. Using $R\cosh(x+\gamma) = R(\cosh x \cosh\gamma + \sinh x \sinh\gamma)$, $R = \sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$ if $B > A > 0$. If $B = A$, then $A \sinh x + B \cosh x = Ae^x$. If $-A < B < A$, the expression can be written as $\mathrm{R}\sinh(x+\gamma)$ with $R = \sqrt{A^2 - B^2}$ and $\gamma = \tanh^{-1}\frac{B}{A}$. If $B = -A$, then $A\sinh x + B\cosh x = -Ae^{-x}$, and if $B < -A$, the expression can be written as $R\cosh(x+\gamma)$ with $R = -\sqrt{B^2 - A^2}$ and $\gamma = \tanh^{-1}\frac{A}{B}$. For part (i), solving simultaneously gives $a \sinh x + b \cosh x = 1$, which gives the desired solutions using the first result of the question. Similarly for part (ii) using the appropriate result, $x = \sinh^{-1}\left(\frac{1}{\sqrt{a^2-b^2}}\right) - \tanh^{-1}\frac{b}{a}$. For (iii), we require that the conditions for (i) give two solutions, i.e. that $b > a$ and $\left(\frac{1}{\sqrt{b^2-a^2}}\right) > 1$, and so $a < b < \sqrt{a^2+1}$, and vice versa, if this applies there are indeed two solutions. For (iv), we require case (i) to give coincident solutions, i.e. $b = \sqrt{a^2+1}$ and hence $x = -\tanh^{-1}\frac{a}{\sqrt{a^2+1}}$, and so $y = \frac{1}{\sqrt{a^2+1}}$. The reverse argument also applies.

7. Considering $(\omega^r)^n$ establishes by the factor theorem that each factor on the LHS is a factor of the RHS, and comparing coefficients of z^n between the two sides establishes that no numerical factor is required. For part (i), representing X_r by ω^r , then there are two cases to consider, P will be represented either by $re^{\frac{\pi i}{n}}$, or $re^{\left(\frac{\pi}{n}+\pi\right)i}$. The product of moduli is the moduli of the product of factors, and the product of the factors can be simplified using the stem and choosing z in turn as the representations of P to give the required result in both cases. Proceeding similarly for *n* odd, the first case yields $|OP|^n + 1$, and the second, $|OP|^n - 1$, if $|OP| \geq 1$, and $1 - |OP|^n$ if $|OP| < 1$. Using the same representations for the X_r in part (ii), and the same technique with the moduli, the stem can be divided by $(z-1)$ to give $(z-\omega)(z-\omega^2) \ldots (z-\omega^{n-1}) = z^{n-1} + z^{n-2} + \cdots + 1$ which then gives the desired result when $z = 1$.

8. The first result in (i) is obtained by the substitution $x = -u$ (followed by a second $u = x$!). Substituting for $f(-x)$ in the initial statement using the result obtained readily leads to $f(x) = x$ which is simply verified. Alternatively, subtracting the result from the initial equation leads to $f(x) = f(-x)$ which substituting gives the required result again. In part (ii), substituting $K(x)$ for x in the equation for $g(x)$ gives an equation for $g\left(\frac{x+1}{x-1}\right)$ which can be substituted in the equation to be solved to give the desired result. Similarly, in part (iii), substituting $\frac{1}{1-x}$ for x gives an equation for $h\left(\frac{1}{1-x}\right)$ and $h\left(\frac{x-1}{x}\right)$, and then repeating this substitution in the equation just obtained gives an equation for $h\left(\frac{x-1}{x}\right)$ and $h(x)$. Adding the given and last equations and subtracting that first found leads to $h(x) = \frac{1}{2} - x$.

9. There are numerous ways to obtain $PX = \frac{2}{\sqrt{3}}a$ via e.g. knowledge of the centroid of a triangle, Pythagoras' theorem, trigonometry or a combination of these, leading to the initial extension result. Pythagoras' theorem can be used to find RX and hence the given tension. The equation of motion in the direction XP combines the tension in PX and the resolved parts of the tensions in the other two springs. Writing the cosine of the angle between RX and PX produced as

$\frac{\frac{1}{\sqrt{3}}a+x}{\sqrt{a^2+\left(\frac{1}{\sqrt{3}}a+x\right)^2}}$, leads to $-\lambda - \frac{3\lambda}{l}x + 2\lambda\left(\frac{1}{\sqrt{3}}a + x\right)\frac{\sqrt{3}}{2a}\left(1 + \frac{\sqrt{3}}{2}\frac{x}{a} + \frac{3x^2}{4a^2}\right)^{\frac{-1}{2}} = m\ddot{x}$ which making an approximation for small x and the binomial expansion leads to $-\frac{3\lambda}{4la}\left(4a - \sqrt{3}l\right)x = m\ddot{x}$, and hence the final result.

10. Resolving along a line of greatest slope initially, bearing in mind the acceleration due to circular motion, gives an expression for the initial tension in the string which can be substituted in the expression obtained for normal contact force obtained by resolving perpendicular to the slope. Requiring a positive normal contact force then gives the desired result. To complete circles, there must be a tension in the string when the particle is at the highest point it can reach on the plane. Conserving energy gives $v^2 = u^2 - 4ag\cos\beta\sin\alpha$ and resolving down the plane yields $T'\cos\beta + mg\sin\alpha = m\frac{v^2}{a\cos\beta}$ resulting in $u^2 > 5ag\cos\beta\sin\alpha$; this combined with the first result will give the final desired result. (The first result can be found elegantly by resolving perpendicularly to the string.)

11. In part (i), expressing the resistance as kv , then the zero acceleration condition gives $k = \frac{P}{16U^2}$. Writing the equation of motion using $a = v\frac{dv}{dx}$, and solving the differential equation by separating variables, the integration gives $X_1 = \left[\frac{m}{k}\left(2U\ln\left(\frac{4U+v}{4U-v}\right) - v\right)\right]_U^{2U}$ which evaluated and rearranged is the required result. Part (ii) follows a similar route, instead expressing the resistance as kv^2 , with $k = \frac{P}{64U^3}$. The same technique for the differential equation gives a slightly simpler integration to yield the result. $\lambda X_1 - \lambda X_2$ can be manipulated to be $\frac{4}{3}\ln 24 - 2\ln 5 - 1$ which can be shown to be positive using the appropriate bounds and so answering part (iii) that X_1 is the larger.

12. Using the binomial distribution, $\mu = 20n$, $\sigma^2 = 16n$, writing $16n \le X \le 24n$ as $|X - 20n| \le 4n$ enables Chebyshev to be applied with $k = \sqrt{n}$ leading to the required result in (i). Similarly, in part (ii), considering a Poisson distribution with mean n , and appreciating that $|X - n| > n$ implies $X > 2n$ in these circumstances, the same value of k as in part (i) with Chebyshev leads to the desired result.

13. Showing that $X - a$ has the same kurtosis as X requires the expectations of $X - a$, $(X - a - \mu + a)^2$, and $(X - a - \mu + a)^4$ to be obtained and substituted. For part (i), the numerator can be obtained by an integration by parts reducing the integral to the one that gives the variance. Expanding T^4 as $\sum\left(Y_r{}^4 + 4Y_r{}^3Y_s + 6Y_r{}^2Y_s{}^2 + 12Y_sY_tY_r{}^2 + 24Y_rY_sY_tY_u\right)$, where the summation is over all values without repetition, and taking the expectation of these terms gives the requested result in part (ii). Defining $Y_i = X_i - \mu$, the kurtosis of Y_i by the first result gives $E\left(Y_i{}^4\right) = (3+\kappa)\sigma^4$ and defining T as in (ii), the kurtosis of $\sum_{i=1}^{n} X_i$ is, using the result of (ii), $\frac{n(3+\kappa)\sigma^4 + 3n(n-1)\sigma^2\sigma^2}{n^2\sigma^4} - 3$ giving the required answer.

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