Section A: Pure Mathematics

1 *In this question, you are not required to justify the accuracy of the approximations.*

(i) Write down the binomial expansion of $\left(1+\dfrac{k}{100}\right)^{\frac{1}{2}}$ in ascending powers of k, up to and including the k^3 term.

(a) Use the value $k = 8$ to find an approximation to five decimal places for $\sqrt{3}$.

(b) By choosing a suitable integer value of k, find an approximation to five decimal places for $\sqrt{6}$.

(ii) By considering the first two terms of the binomial expansion of $\left(1+\dfrac{k}{1000}\right)^{\frac{1}{3}}$, show that $\dfrac{3029}{2100}$ is an approximation to $\sqrt[3]{3}$.

2 A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q - p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q-p)^4$.

3 By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}$.

(i) Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x$.

(a) Evaluate I.

(b) Use the substitution $t = \tan\frac{1}{2}x$ to show that $\displaystyle\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t = \frac{1}{2}I$.

(ii) Evaluate $\displaystyle\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,\mathrm{d}t$.

4 Given that $\cos A$, $\cos B$ and β are non-zero, show that the equation

$$\alpha \sin(A-B) + \beta \cos(A+B) = \gamma \sin(A+B)$$

reduces to the form

$$(\tan A - m)(\tan B - n) = 0\,,$$

where m and n are independent of A and B, if and only if $\alpha^2 = \beta^2 + \gamma^2$.

Determine all values of x, in the range $0 \leqslant x < 2\pi$, for which:

(i) $2\sin(x - \frac{1}{4}\pi) + \sqrt{3}\cos(x + \frac{1}{4}\pi) = \sin(x + \frac{1}{4}\pi)\,;$

(ii) $2\sin(x - \frac{1}{6}\pi) + \sqrt{3}\cos(x + \frac{1}{6}\pi) = \sin(x + \frac{1}{6}\pi)\,;$

(iii) $2\sin(x + \frac{1}{3}\pi) + \sqrt{3}\cos(3x) = \sin(3x)\,.$

5 In this question, $\mathrm{f}^2(x)$ denotes $\mathrm{f}(\mathrm{f}(x))$, $\mathrm{f}^3(x)$ denotes $\mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$, and so on.

(i) The function f is defined, for $x \neq \pm 1/\sqrt{3}$, by

$$\mathrm{f}(x) = \frac{x + \sqrt{3}}{1 - \sqrt{3}\,x}\,.$$

Find by direct calculation $\mathrm{f}^2(x)$ and $\mathrm{f}^3(x)$, and determine $\mathrm{f}^{2007}(x)$.

(ii) Show that $\mathrm{f}^n(x) = \tan(\theta + \frac{1}{3}n\pi)$, where $x = \tan\theta$ and n is any positive integer.

(iii) The function $\mathrm{g}(t)$ is defined, for $|t| \leqslant 1$ by $\mathrm{g}(t) = \frac{\sqrt{3}}{2}t + \frac{1}{2}\sqrt{1-t^2}$. Find an expression for $\mathrm{g}^n(t)$ for any positive integer n.

6 **(i)** Differentiate $\ln\left(x + \sqrt{3 + x^2}\,\right)$ and $x\sqrt{3 + x^2}$ and simplify your answers.

Hence find $\int \sqrt{3 + x^2}\,\mathrm{d}x$.

(ii) Find the two solutions of the differential equation

$$3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

that satisfy $y = 0$ when $x = 1$.

7 A function $\mathrm{f}(x)$ is said to be concave on some interval if $\mathrm{f}''(x) < 0$ in that interval. Show that $\sin x$ is concave for $0 < x < \pi$ and that $\ln x$ is concave for $x > 0$.

Let $\mathrm{f}(x)$ be concave on a given interval and let $x_1, x_2, \ldots, x_n$ lie in the interval. *Jensen's inequality* states that

$$\frac{1}{n}\sum_{k=1}^{n} \mathrm{f}(x_k) \leqslant \mathrm{f}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_n$. You may use this result without proving it.

(i) Given that A, B and C are angles of a triangle, show that

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}\,.$$

(ii) By choosing a suitable function f, prove that

$$\sqrt[n]{t_1 t_2 \cdots t_n} \;\leqslant\; \frac{t_1 + t_2 + \cdots + t_n}{n}$$

for any positive integer n and for any positive numbers $t_1, t_2, \ldots, t_n$.

Hence:

(a) show that $x^4 + y^4 + z^4 + 16 \geqslant 8xyz$, where x, y and z are any positive numbers;

(b) find the minimum value of $x^5 + y^5 + z^5 - 5xyz$, where x, y and z are any positive numbers.

8 The points B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$, respectively, relative to the origin A, and A, B and C are not collinear.

(i) The point X has position vector $s\mathbf{b} + t\mathbf{c}$. Describe the locus of X when $s + t = 1$.

(ii) The point P has position vector $\beta\mathbf{b} + \gamma\mathbf{c}$, where β and γ are non-zero, and $\beta + \gamma \neq 1$. The line AP cuts the line BC at D. Show that $BD : DC = \gamma : \beta$.

(iii) The line BP cuts the line CA at E, and the line CP cuts the line AB at F. Show that

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1\,.$$

Section B: Mechanics

9 A solid right circular cone, of mass M, has semi-vertical angle α and smooth surfaces. It stands with its base on a smooth horizontal table. A particle of mass m is projected so that it strikes the curved surface of the cone at speed u. The coefficient of restitution between the particle and the cone is e. The impact has no rotational effect on the cone and the cone has no vertical velocity after the impact.

(i) The particle strikes the cone in the direction of the normal at the point of impact. Explain why the trajectory of the particle immediately after the impact is parallel to the normal to the surface of the cone. Find an expression, in terms of M, m, α, e and u, for the speed at which the cone slides along the table immediately after impact.

(ii) If instead the particle falls vertically onto the cone, show that the speed w at which the cone slides along the table immediately after impact is given by

$$w = \frac{mu(1+e)\sin\alpha\cos\alpha}{M + m\cos^2\alpha}\,.$$

Show also that the value of α for which w is greatest is given by

$$\cos\alpha = \sqrt{\frac{M}{2M+m}}\,.$$

10 A solid figure is composed of a uniform solid cylinder of density ρ and a uniform solid hemisphere of density 3ρ. The cylinder has circular cross-section, with radius r, and height $3r$, and the hemisphere has radius r. The flat face of the hemisphere is joined to one end of the cylinder, so that their centres coincide.

The figure is held in equilibrium by a force P so that one point of its flat base is in contact with a rough horizontal plane and its base is inclined at an angle α to the horizontal. The force P is horizontal and acts through the highest point of the base. The coefficient of friction between the solid and the plane is μ. Show that

$$\mu \geqslant \left|\tfrac{9}{8} - \tfrac{1}{2}\cot\alpha\right|\,.$$

11 *In this question take the acceleration due to gravity to be* $10\,\mathrm{m\,s^{-2}}$ *and neglect air resistance.*

The point O lies in a horizontal field. The point B lies $50\,\mathrm{m}$ east of O. A particle is projected from B at speed $25\,\mathrm{m\,s^{-1}}$ at an angle $\arctan\frac{1}{2}$ above the horizontal and in a direction that makes an angle $60°$ with OB; it passes to the north of O.

(i) Taking unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to O at time t seconds after the particle was projected, and show that its distance from O is

$$5(t^2 - \sqrt{5}t + 10)\,\mathrm{m}.$$

When this distance is shortest, the particle is at point P. Find the position vector of P and its horizontal bearing from O.

(ii) Show that the particle reaches its maximum height at P.

(iii) When the particle is at P, a marksman fires a bullet from O directly at P. The initial speed of the bullet is $350\,\mathrm{m\,s^{-1}}$. Ignoring the effect of gravity on the bullet show that, when it passes through P, the distance between P and the particle is approximately $3\,\mathrm{m}$.

Section C: Probability and Statistics

12 I have two identical dice. When I throw either one of them, the probability of it showing a 6 is p and the probability of it not showing a 6 is q, where $p + q = 1$. As an experiment to determine p, I throw the dice simultaneously until at least one die shows a 6. If both dice show a six on this throw, I stop. If just one die shows a six, I throw the other die until it shows a 6 and then stop.

(i) Show that the probability that I stop after r throws is $pq^{r-1}(2 - q^{r-1} - q^r)$, and find an expression for the expected number of throws.

[**Note:** You may use the result $\sum\limits_{r=0}^{\infty} rx^r = x(1-x)^{-2}$.]

(ii) In a large number of such experiments, the mean number of throws was m. Find an estimate for p in terms of m.

13 Given that $0 < r < n$ and r is much smaller than n, show that $\dfrac{n-r}{n} \approx \mathrm{e}^{-r/n}$.

There are k guests at a party. Assuming that there are exactly 365 days in the year, and that the birthday of any guest is equally likely to fall on any of these days, show that the probability that there are at least two guests with the same birthday is approximately $1 - \mathrm{e}^{-k(k-1)/730}$.

Using the approximation $\frac{253}{365} \approx \ln 2$, find the smallest value of k such that the probability that at least two guests share the same birthday is at least $\frac{1}{2}$.

How many guests must there be at the party for the probability that at least one guest has the same birthday as the host to be at least $\frac{1}{2}$?

14 The random variable X has a continuous probability density function $\mathrm{f}(x)$ given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ \ln x & \text{for } 1 \leqslant x \leqslant k \\ \ln k & \text{for } k \leqslant x \leqslant 2k \\ a - bx & \text{for } 2k \leqslant x \leqslant 4k \\ 0 & \text{for } x \geqslant 4k \end{cases}$$

where k, a and b are constants.

(i) Sketch the graph of $y = \mathrm{f}(x)$.

(ii) Determine a and b in terms of k and find the numerical values of k, a and b.

(iii) Find the median value of X.