SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board
on behalf of the Cambridge Colleges

9470

MATHEMATICS II

Thursday 27 June 1996, morning

3 hours

Additional materials:
script paper; graph paper; MF(STEP)1.
To be brought by candidate: electronic calculator; standard geometrical instruments.

All questions carry equal weight.

You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

You may attempt as many questions as you wish with no restriction of choice but marks will be assessed on the **six** *questions best answered.*

You are provided with Mathematical Formulae and Tables MF(STEP)1.

The use of electronic calculators is permitted.

This Question Paper consists of 7 printed pages and 1 blank page.

Section A: Pure Mathematics

1 **(i)** Find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4)^3.$$

You should set out your working clearly.

(ii) By considering the binomial expansions of $(1+x)^{-2}$ and $(1+x)^{-6}$, or otherwise, find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^3.$$

2 Consider the system of equations

$$\begin{aligned} 2yz + zx - 5xy &= 2 \\ yz - zx + 2xy &= 1 \\ yz - 2zx + 6xy &= 3. \end{aligned}$$

Show that

$$xyz = \pm 6$$

and find the possible values of x, y and z.

3 The Fibonacci numbers F_n are defined by the conditions $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = F_n + F_{n-1}$$

for all $n \geqslant 1$. Show that $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and compute F_5, F_6 and F_7.

Compute $F_{n+1}F_{n-1} - F_n^2$ for a few values of n; guess a general formula and prove it by induction, or otherwise.

By induction on k, or otherwise, show that

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

for all positive integers n and k.

4 Show that $\cos 4u = 8\cos^4 u - 8\cos^2 u + 1$.

If

$$I = \int_{-1}^{1} \frac{1}{\sqrt{(1+x)} + \sqrt{(1-x)} + 2}\, \mathrm{d}x,$$

show, by using the change of variable $x = \cos t$, that

$$I = \int_0^{\pi} \frac{\sin t}{4\cos^2\left(\frac{t}{4} - \frac{\pi}{8}\right)}\, \mathrm{d}t.$$

By using the further change of variable $u = \frac{t}{4} - \frac{\pi}{8}$, or otherwise, show that

$$I = 4\sqrt{2} - \pi - 2.$$

[You may assume that $\tan\frac{\pi}{8} = \sqrt{2} - 1$.]

5 If

$$z^4 + z^3 + z^2 + z + 1 = 0 \qquad (*)$$

and $u = z + z^{-1}$, find the possible values of u. Hence find the possible values of z. [Do not try to simplify your answers.]

Show that, if z satisfies $(*)$, then

$$z^5 - 1 = 0.$$

Hence write the solutions of $(*)$ in the form $z = r(\cos\theta + i\sin\theta)$ for suitable real r and θ. Deduce that

$$\sin\frac{2\pi}{5} = \frac{\sqrt{(10 + 2\sqrt{5})}}{4} \quad \text{and} \quad \cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}.$$

6 A *proper factor* of a positive integer N is an integer M, with $M \neq 1$ and $M \neq N$, which divides N without remainder. Show that 12 has 4 proper factors and 16 has 3.

Suppose that N has the prime factorisation

$$N = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r},$$

where $p_1, p_2, \dots, p_r$ are distinct primes and $m_1, m_2, \dots, m_r$ are positive integers. How many proper factors does N have and why?

Find:

(i) the smallest positive integer which has precisely 12 proper factors;

(ii) the smallest positive integer which has at least 12 proper factors.

7 Consider a fixed square $ABCD$ and a variable point P in the plane of the square. We write the perpendicular distance from P to AB as p, from P to BC as q, from P to CD as r and from P to DA as s. (Remember that distance is never negative, so $p, q, r, s \geqslant 0$.) If $pr = qs$, show that the only possible positions of P lie on two straight lines and a circle and that every point on these two lines and a circle is indeed a possible position of P.

8 Suppose that

$$\mathrm{f}''(x) + \mathrm{f}(-x) = x + 3\cos 2x$$

and $\mathrm{f}(0) = 1$, $\mathrm{f}'(0) = -1$. If $\mathrm{g}(x) = \mathrm{f}(x) + \mathrm{f}(-x)$, find $\mathrm{g}(0)$ and show that $\mathrm{g}'(0) = 0$. Show that

$$\mathrm{g}''(x) + \mathrm{g}(x) = 6\cos 2x,$$

and hence find $\mathrm{g}(x)$.

Similarly, if $\mathrm{h}(x) = \mathrm{f}(x) - \mathrm{f}(-x)$, find $\mathrm{h}(x)$ and show that

$$\mathrm{f}(x) = 2\cos x - \cos 2x - x.$$

Section B: Mechanics

9 A child's toy consists of a solid cone of height λa and a solid hemisphere of radius a, made out of the same uniform material and fastened together so that their plane faces coincide. (Thus the diameter of the hemisphere is equal to that of the base of the cone.) Show that if $\lambda < \sqrt{3}$ the toy will always move to an upright position if placed with the surface of the hemisphere on a horizontal table, but that if $\lambda > \sqrt{3}$ the toy may overbalance.

Show, however, that if the toy is placed with the surface of the cone touching the table it will remain there whatever the value of λ.

[The centre of gravity of a uniform solid cone of height h is a height $h/4$ above its base. The centre of gravity of a uniform solid hemisphere of radius a is at distance $3a/8$ from the centre of its base.]

10 The plot of 'Rhode Island Red and the Henhouse of Doom' calls for the heroine to cling on to the circumference of a fairground wheel of radius a rotating with constant angular velocity ω about its horizontal axis and then let go. Let ω_0 be the largest value of ω for which it is not possible for her subsequent path to carry her higher than the top of the wheel. Find ω_0 in terms of a and g.

If $\omega > \omega_0$ show that the greatest height above the top of the wheel to which she can rise is

$$\frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

11 A particle hangs in equilibrium from the ceiling of a stationary lift, to which it is attached by an elastic string of natural length l extended to a length $l + a$. The lift now descends with constant acceleration f such that $0 < f < g/2$. Show that the extension y of the string from its equilibrium length satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{g}{a}y = g - f.$$

Hence show that the string never becomes slack and the amplitude of the oscillation of the particle is af/g.

After a time T the lift stops accelerating and moves with constant velocity. Show that the string never becomes slack and the amplitude of the oscillation is now

$$\frac{2af}{g}|\sin\tfrac{1}{2}\omega T|,$$

where $\omega^2 = g/a$.

Section C: Probability and Statistics

12 **(i)** Let X_1, X_2, ..., X_n be independent random variables each of which is uniformly distributed on $[0, 1]$. Let Y be the largest of X_1, X_2, ..., X_n. By using the fact that $Y < \lambda$ if and only if $X_j < \lambda$ for $1 \leqslant j \leqslant n$, find the probability density function of Y. Show that the variance of Y is

$$\frac{n}{(n+2)(n+1)^2}.$$

(ii) The probability that a neon light switched on at time 0 will have failed by a time $t > 0$ is $1 - e^{-t/\lambda}$ where $\lambda > 0$. I switch on n independent neon lights at time zero. Show that the expected time until the first failure is λ/n.

13 By considering the coefficients of t^n in the equation

$$(1+t)^n(1+t)^n = (1+t)^{2n},$$

or otherwise, show that

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{r}\binom{n}{n-r} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n}.$$

The large American city of Triposville is laid out in a square grid with equally spaced streets running east-west and avenues running north-south. My friend is staying at a hotel n avenues west and n streets north of my hotel. Both hotels are at intersections. We set out from our own hotels at the same time. We walk at the same speed, taking 1 minute to go from one intersection to the next. Every time I reach an intersection I go north with probability 1/2 or west with probability 1/2. Every time my friend reaches an intersection she goes south with probability 1/2 or east with probability 1/2. Our choices are independent of each other and of our previous decisions. Indicate by a sketch or by a brief description the set of points where we could meet. Find the probability that we meet.

Suppose that I oversleep and leave my hotel $2k$ minutes later than my friend leaves hers, where k is an integer and $0 \leqslant 2k \leqslant n$. Find the probability that we meet. Have you any comment? If $n = 1$ and I leave my hotel 1 minute later than my friend leaves hers, what is the probability that we meet and why?

14 The random variable X is uniformly distributed on $[0,1]$. A new random variable Y is defined by the rule

$$Y = \begin{cases} 1/4 & \text{if } X \leqslant 1/4\ , \\ X & \text{if } 1/4 \leqslant X \leqslant 3/4\ , \\ 3/4 & \text{if } X \geqslant 3/4\ . \end{cases}$$

Find $\mathrm{E}(Y^n)$ for all integers $n \geqslant 1$.

Show that $\mathrm{E}(Y) = \mathrm{E}(X)$ and that

$$\mathrm{E}(X^2) - \mathrm{E}(Y^2) = \frac{1}{24}.$$

By using the fact that $4^n = (3+1)^n$, or otherwise, show that $\mathrm{E}(X^n) > \mathrm{E}(Y^n)$ for $n \geqslant 2$.

Suppose that Y_1, Y_2, ... are independent random variables each having the same distribution as Y. Find, to a good approximation, K such that

$$\mathrm{P}(Y_1 + Y_2 + \cdots + Y_{240000} < K) = 3/4.$$