SIXTH TERM EXAMINATION PAPERS

administered by the Oxford and Cambridge Schools Examination Board on behalf of the Cambridge Colleges

9475

MATHEMATICS III

Friday 30 June 1995, afternoon

3 hours

Additional materials:
script paper; graph paper; MF(STEP)1.
To be brought by candidate: electronic calculator; standard geometrical instruments.

All questions carry equal weight.

You are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.

Candidates may attempt as many questions as you wish but marks will be assessed on the **six** *questions best answered.*

Mathematical Formulae and Tables (MFSTEP) are provided.

Electronic calculators may be used.

This Question Paper consists of 7 printed pages and 1 blank page.

Section A: Pure Mathematics

1 Find the simultaneous solutions of the three linear equations

$$\begin{aligned} a^2x + ay + z &= a^2 \\ ax + y + bz &= 1 \\ a^2bx + y + bz &= b \end{aligned}$$

for all possible real values of a and b.

2 If

$$\mathrm{I}_n = \int_0^a x^{n+\frac{1}{2}}(a-x)^{\frac{1}{2}}\,\mathrm{d}x,$$

show that $\mathrm{I}_0 = \pi a^2/8$.

Show that $(2n+4)\,\mathrm{I}_n = (2n+1)\,a\,\mathrm{I}_{n-1}$ and hence evaluate I_n.

3 What is the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2k\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

for each of the cases: (i) $k > 1$; (ii) $k = 1$; (iii) $0 < k < 1$?

In case (iii) the equation represents damped simple harmonic motion with damping factor k. Let $x(0) = 0$ and let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of successive maxima and minima, so that if x_n is a maximum then x_{n+1} is the next minimum. Show that $|x_{n+1}/x_n|$ takes a value α which is independent of n, and that

$$k^2 = \frac{(\ln\alpha)^2}{\pi^2 + (\ln\alpha)^2}.$$

4 Let

$$C_n(\theta) = \sum_{k=0}^{n} \cos k\theta$$

and

$$S_n(\theta) = \sum_{k=0}^{n} \sin k\theta,$$

where n is a positive integer and $0 < \theta < 2\pi$. Show that

$$C_n(\theta) = \frac{\cos\left(\frac{1}{2}n\theta\right)\sin\left(\frac{1}{2}(n+1)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)},$$

and obtain the corresponding expression for $S_n(\theta)$.

Hence, or otherwise, show that for $0 < \theta < 2\pi$,

$$\left|C_n(\theta) - \frac{1}{2}\right| \leqslant \frac{1}{2\sin(\frac{1}{2}\theta)}.$$

5 Show that $y = \sin^2(m \sin^{-1} x)$ satisfies the differential equation

$$(1 - x^2)y^{(2)} = xy^{(1)} + 2m^2(1 - 2y),$$

and deduce that, for all $n \geq 1$,

$$(1 - x^2)y^{(n+2)} = (2n + 1)xy^{(n+1)} + (n^2 - 4m^2)y^{(n)},$$

where $y^{(n)}$ denotes the nth derivative of y.

Derive the Maclaurin series for y, making it clear what the general term is.

6 The variable non-zero complex number z is such that

$$|z - i| = 1.$$

Find the modulus of z when its argument is θ. Find also the modulus and argument of $1/z$ in terms of θ and show in an Argand diagram the loci of points which represent z and $1/z$.

Find the locus C in the Argand diagram such that $w \in C$ if, and only if, the real part of $(1/w)$ is -1.

7 Consider the following sets with the usual definition of multiplication appropriate to each. In each case you may assume that the multiplication is associative. In each case state, giving adequate reasons, whether or not the set is a group:

(i) the complex numbers of unit modulus;

(ii) the integers modulo 4;

(iii) the matrices

$$\mathrm{M}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 \leqslant \theta < 2\pi$;

(iv) the integers 1, 3, 5, 7 modulo 8;

(v) the 2×2 matrices all of whose entries are integers;

(vi) the integers 1, 2, 3, 4 modulo 5.

In the case of each pair of groups above state, with reasons, whether or not they are isomorphic.

8 A plane π in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r} \cdot \mathbf{n_i} = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

Section B: Mechanics

9 A thin circular disc of mass m, radius r and with its centre of mass at its centre C can rotate freely in a vertical plane about a fixed horizontal axis through a point O of its circumference. A particle P, also of mass m, is attached to the circumference of the disc so that the angle OCP is 2α, where $\alpha \leqslant \pi/2$.

(i) In the position of stable equilibrium OC makes an angle β with the vertical. Prove that

$$\tan\beta = \frac{\sin 2\alpha}{2 - \cos 2\alpha}.$$

(ii) The density of the disc at a point distant x from C is $\rho x/r$. Show that its moment of inertia about the horizontal axis through O is $8mr^2/5$.

(iii) The mid-point of CP is Q. The disc is held at rest with OQ horizontal and C lower than P and it is then released. Show that the speed v with which C is moving when P passes vertically below O is given by

$$v^2 = \frac{15gr\sin\alpha}{2(2+5\sin^2\alpha)}.$$

Find the maximum value of v^2 as α is varied.

10 A cannon is situated at the bottom of a plane inclined at angle β to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed u. Ignoring air resistance, find the angle of firing which will maximize the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1+\sin\beta)}$$

from the cannon.

11 A ship is sailing due west at V knots while a plane, with an airspeed of kV knots, where $k > \sqrt{2}$, patrols so that it is always to the north west of the ship. If the wind in the area is blowing from north to south at V knots and the pilot is instructed to return to the ship every thirty minutes, how long will her outward flight last?

Assume that the maximum distance of the plane from the ship during the above patrol was d_w miles. If the air now becomes dead calm, and the pilot's orders are maintained, show that the ratio d_w/d_c of d_w to the new maximum distance, d_c miles, of the plane from the ship is

$$\frac{k^2-2}{2k(k^2-1)}\sqrt{4k^2-2}.$$

Section C: Probability & Statistics

12 The random variables X and Y are independently normally distributed with means 0 and variances 1. Show that the joint probablility density function for (X, Y) is

$$\mathrm{f}\,(x,y) = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \qquad -\infty < x < \infty,\ -\infty < y < \infty.$$

If (x, y) are the coordinates, referred to rectangular axes, of a point in the plane, explain what is meant by saying that this density is radially symmetrical.

The random variables U and V have a joint probability density function which is radially symmetrical (in the above sense). By considering the straight line with equation $U = kV$, or otherwise, show that

$$\mathbf{P}\left(\frac{U}{V} < k\right) = 2\,\mathbf{P}(U < kV\,,\ V > 0).$$

Hence, or otherwise, show that the probability density function of U/V is

$$\mathrm{g}\,(k) = \frac{1}{\pi(1+k^2)} \qquad -\infty < k < \infty.$$

13 A message of 10^k binary digits is sent along a fibre optic cable with high probabilities p_0 and p_1 that the digits 0 and 1, respectively, are received correctly. If the probability of a digit in the original message being a 1 is α, find the probability that the entire message is received correctly.

Find the probability β that a randomly chosen digit in the message is received as a 1 and show that $\beta = \alpha$ if, and only if,

$$\alpha = \frac{q_0}{q_1 + q_0},$$

where $q_0 = 1 - p_0$ and $q_1 = 1 - p_1$. If this condition is satisfied and the received message consists entirely of zeros, what is the probability that it is correct?

If now $q_0 = q_1 = q$ and $\alpha = \frac{1}{2}$, find the approximate value of q which will ensure that a message of one million binary digits has a fifty-fifty chance of being received entirely correctly.

The probability of error q is proportional to the square of the length of the cable. Initially the length is such that the probability of a message of one million binary bits, among which 0 and 1 are equally likely, being received correctly is $\frac{1}{2}$. What would this probability become if a booster station were installed at its mid-point, assuming that the booster station re-transmits the received version of the message, and assuming that terms of order q^2 may be ignored?

14 A candidate finishes examination questions in time T, where T has probability density function

$$\mathrm{f}(t) = te^{-t} \qquad t \geqslant 0,$$

the probabilities for the various questions being independent. Find the moment generating function of T and hence find the moment generating function for the total time U taken to finish two such questions. Show that the probability density function for U is

$$\mathrm{g}(u) = \frac{1}{6}u^3e^{-u} \qquad u \geqslant 0.$$

Find the probability density function for the total time taken to answer n such questions.