

**SIXTH TERM EXAMINATION PAPER**

administered by the Oxford and Cambridge Schools Examination Board  
on behalf of the Cambridge Colleges

9475

**FURTHER MATHEMATICS**

**PAPER B**

Friday 28 June 1991, afternoon

3 hours

*All questions carry equal weight.*

*Candidates are reminded that extra credit is given for complete answers and that little credit is given for isolated fragments.*

*Candidates may attempt as many questions as they wish but marks will be assessed on the six questions best answered.*

*Mathematical Formulae and tables MF(STEP 1) are provided.*

*Electronic calculators may be used.*

1 (a) Evaluate

$$\sum_{r=1}^n \frac{6}{r(r+1)(r+3)}.$$

(b) Expand  $\ln(1+x+x^2+x^3)$  as a series in powers of  $x$ , where  $|x| < 1$ , giving the first five non-zero terms and the general term.

(c) Expand  $e^{x \ln(1+x)}$  as a series in powers of  $x$ , where  $-1 < x \leq 1$ , as far as the term in  $x^4$ .

2 The distinct points  $P_1, P_2, P_3, Q_1, Q_2$  and  $Q_3$  in the Argand diagram are represented by the complex numbers  $z_1, z_2, z_3, w_1, w_2$  and  $w_3$  respectively. Show that the triangles  $P_1P_2P_3$  and  $Q_1Q_2Q_3$  are similar, with  $P_i$  corresponding to  $Q_i$  ( $i = 1, 2, 3$ ) and the rotation from 1 to 2 to 3 being in the same sense for both triangles, if and only if

$$\frac{z_1 - z_2}{z_1 - z_3} = \frac{w_1 - w_2}{w_1 - w_3}.$$

Verify that this condition may be written

$$\det \begin{pmatrix} z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

(i) Show that if  $w_i = z_i^2$  ( $i = 1, 2, 3$ ) then triangle  $P_1P_2P_3$  is not similar to triangle  $Q_1Q_2Q_3$ .

(ii) Show that if  $w_i = z_i^3$  ( $i = 1, 2, 3$ ) then triangle  $P_1P_2P_3$  is similar to triangle  $Q_1Q_2Q_3$  if and only if the centroid of triangle  $P_1P_2P_3$  is the origin. [The *centroid* of triangle  $P_1P_2P_3$  is represented by the complex number  $\frac{1}{3}(z_1 + z_2 + z_3)$ .]

(iii) Show that the triangle  $P_1P_2P_3$  is equilateral if and only if

$$z_2z_3 + z_3z_1 + z_1z_2 = z_1^2 + z_2^2 + z_3^2.$$

3 The function  $f$  is defined for  $x < 2$  by

$$f(x) = 2|x^2 - x| + |x^2 - 1| - 2|x^2 + x|.$$

Find the maximum and minimum points and the points of inflection of the graph of  $f$  and sketch this graph. Is  $f$  continuous everywhere? Is  $f$  differentiable everywhere?

Find the inverse of the function  $f$ , i.e. expressions for  $f^{-1}(x)$ , defined in the various appropriate intervals.

4 The point  $P$  moves on a straight line in three-dimensional space. The position of  $P$  is observed from the points  $O_1(0, 0, 0)$  and  $O_2(8a, 0, 0)$ . At times  $t = t_1$  and  $t = t'_1$ , the lines of sight from  $O_1$  are along the lines  $\frac{x}{2} = \frac{z}{3}$ ,  $y = 0$  and  $x = 0$ ,  $\frac{y}{3} = \frac{z}{4}$  respectively. At times  $t = t_2$  and  $t = t'_2$ , the lines of sight from  $O_2$  are  $\frac{x-8a}{-3} = \frac{y}{1} = \frac{z}{3}$  and  $\frac{x-8a}{-4} = \frac{y}{2} = \frac{z}{5}$  respectively. Find an equation or equations for the path of  $P$ .

5 The curve  $C$  has the differential equation in polar coordinates

$$\frac{d^2 r}{d\theta^2} + 4r = 5 \sin 3\theta, \quad \text{for } \frac{\pi}{5} \leq \theta \leq \frac{3\pi}{5},$$

and, when  $\theta = \frac{\pi}{2}$ ,  $r = 1$  and  $\frac{dr}{d\theta} = -2$ . Show that  $C$  forms a closed loop and that the area of the region enclosed by  $C$  is

$$\frac{\pi}{5} + \frac{25}{48} \left[ \sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right) \right].$$

6 The transformation  $T$  from  $\begin{pmatrix} x \\ y \end{pmatrix}$  to  $\begin{pmatrix} x' \\ y' \end{pmatrix}$  in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where  $u$  is a positive real constant. Show that the curve with equation  $x^2 - y^2 = 1$  is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves.

A line, not necessarily through the origin, which has gradient  $\tanh v$  transforms under  $T$  into a line with gradient  $\tanh v'$ . Show that  $v' = v + u$ .

The lines  $\ell_1$  and  $\ell_2$  with gradients  $\tanh v_1$  and  $\tanh v_2$  transform under  $T$  into lines with gradients  $\tanh v'_1$  and  $\tanh v'_2$  respectively. Find the relation satisfied by  $v_1$  and  $v_2$  that is the necessary and sufficient for  $\ell_1$  and  $\ell_2$  to intersect at the same angle as their transforms.

In the case when  $\ell_1$  and  $\ell_2$  meet at the origin, illustrate in a diagram the relation between  $\ell_1$ ,  $\ell_2$  and their transforms.

7 (a) Prove that

$$\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx = \frac{1}{2} \int_0^{\pi/2} \ln \sin 2x \, dx - \frac{1}{4} \pi \ln 2.$$

and

$$\int_0^{\pi/2} \ln \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = \int_0^{\pi/2} \ln \sin x \, dx.$$

Hence, or otherwise, evaluate  $\int_0^{\pi/2} \ln \sin x \, dx$ .

[You may assume that all the integrals converge.]

(b) Given that  $\ln u < u$  for  $u \geq 1$  deduce that

$$\frac{1}{2} \ln x < \sqrt{x} \quad \text{for} \quad x \geq 1.$$

Deduce that  $\frac{\ln x}{x} \rightarrow 0$  as  $x \rightarrow \infty$  and that  $x \ln x \rightarrow 0$  as  $x \rightarrow 0$  through positive values.

(c) Using the results of parts (a) and (b), or otherwise, evaluate  $\int_0^{\pi/2} x \cot x \, dx$ .

8 (a) The integral  $I_k$  is defined by

$$I_k = \int_0^{\theta} \cos^k x \cos kx \, dx.$$

Prove that  $2kI_k = kI_{k-1} + \cos^k \theta \sin k\theta$ .

(b) Prove that

$$1 + m \cos 2\theta + \binom{m}{2} \cos 4\theta + \cdots + \binom{m}{r} \cos 2r\theta + \cdots + \cos 2m\theta = 2^m \cos^m \theta \cos m\theta.$$

(c) Using the results of parts (a) and (b), show that

$$\begin{aligned} m \frac{\sin 2\theta}{2} + \binom{m}{2} \frac{\sin 4\theta}{4} + \cdots + \binom{m}{r} \frac{\sin 2r\theta}{2r} + \cdots + \frac{\sin 2m\theta}{2m} \\ = \cos \theta \sin \theta + \cos^2 \theta \sin 2\theta + \cdots + \frac{1}{r} 2^{r-1} \cos^r \theta \sin r\theta + \cdots + \frac{1}{m} 2^{m-1} \cos^m \theta \sin m\theta. \end{aligned}$$

9 The parametric equations  $E_1$  and  $E_2$  define the same ellipse, in terms of the parameters  $\theta_1$  and  $\theta_2$ , (though not referred to the same coordinate axes).

$$\begin{aligned} E_1 : \quad & x = a \cos \theta_1, & y &= b \sin \theta_1, \\ E_2 : \quad & x = \frac{k \cos \theta_2}{1 + e \cos \theta_2}, & y &= \frac{k \sin \theta_2}{1 + e \cos \theta_2}, \end{aligned}$$

where  $0 < b < a$ ,  $0 < e < 1$  and  $0 < k$ . Find the position of the axes for  $E_2$  relative to the axes for  $E_1$  and show that  $k = a(1 - e^2)$  and  $b^2 = a^2(1 - e^2)$ .

[The standard polar equation of an ellipse is  $r = \frac{\ell}{1 + e \cos \theta}$ .]

By considering expressions for the length of the perimeter of the ellipse, or otherwise, prove that

$$\int_0^\pi \sqrt{1 - e^2 \cos^2 \theta} d\theta = \int_0^\pi \frac{1 - e^2}{(1 + e \cos \theta)^2} \sqrt{1 + e^2 + 2e \cos \theta} d\theta.$$

Given that  $e$  is so small that  $e^6$  may be neglected, show that the value of either integral is

$$\frac{1}{64} \pi (64 - 16e^2 - 3e^4).$$

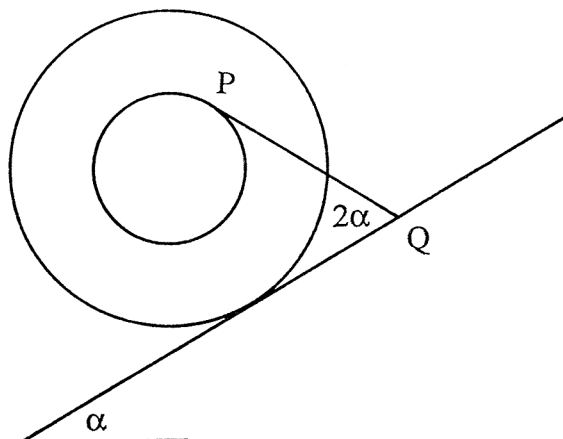
10 The equation  $x^n - qx^{n-1} + r = 0$ , where  $n \geq 5$  and  $q$  and  $r$  are real constants, has roots  $\alpha_1, \alpha_2, \dots, \alpha_n$ . The sum of the products of  $m$  distinct roots is denoted by  $\Sigma_m$  (so that, for example,  $\Sigma_3 = \sum \alpha_i \alpha_j \alpha_k$  where the sum runs over all values of  $i, j$  and  $k$  with  $n \geq i > j > k \geq 1$ ). The sum of the  $m$ th powers of the roots is denoted by  $S_m$  (so that, for example,  $S_3 = \sum_{i=1}^n \alpha_i^3$ ). Prove that  $S_p = q^p$  for  $1 \leq p \leq n - 1$ .

[You may assume that for any  $n$ th degree equation and  $1 \leq p \leq n$

$$S_p - S_{p-1} \Sigma_1 + S_{p-2} \Sigma_2 - \dots + (-1)^{p-1} S_1 \Sigma_{p-1} + (-1)^p p \Sigma_p = 0.]$$

Find expressions for  $S_n, S_{n+1}$  and  $S_{n+2}$  in terms of  $q, r$  and  $n$ . Suggest an expression for  $S_{n+m}$ , where  $m < n$ , and prove its validity by induction.

11



A uniform circular cylinder of radius  $2a$  with a groove of radius  $a$  cut in its central cross-section has mass  $M$ . It rests, as shown in the diagram, on a rough plane inclined at an acute angle  $\alpha$  to the horizontal. It is supported by a light inextensible string wound round the groove and attached to the cylinder at one end. The other end of the string is attached to the plane at  $Q$ , the free part of the string,  $PQ$ , making an angle  $2\alpha$  with the inclined plane. The coefficient of friction at the contact between the cylinder and the plane is  $\mu$ . Show that  $\mu \geq \frac{1}{3} \tan \alpha$ .

The string  $PQ$  is now detached from the plane and the end  $Q$  is fastened to a particle of mass  $3M$  which is placed on the plane, the position of the string remaining unchanged. Given that  $\tan \alpha = \frac{1}{2}$  and that the system remains in equilibrium, find the least value of the coefficient of friction between the particle and the plane.

12 A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a \cos \theta \quad y = b \sin \theta$$

where the  $x$ -axis is horizontal and the  $y$ -axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section. By first finding  $\ddot{x}$  and  $\ddot{y}$ , or otherwise, show that the acceleration along the inward normal at the point with parameter  $\theta$  is

$$\frac{ab \dot{\theta}^2}{\sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}}$$

The particle is projected along the surface in the vertical cross-section plane, with speed  $2\sqrt{bg}$ , from the lowest point. Given that  $2a = 3b$ , show that it will leave the surface at the point with parameter  $\theta$  where

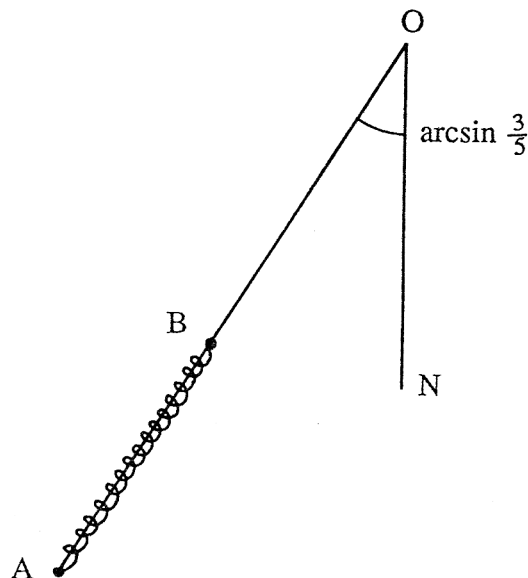
$$5 \sin^3 \theta + 12 \sin \theta - 8 = 0.$$

13 A smooth particle  $P_1$  is projected from a point  $O$  on the horizontal floor of a room which has a horizontal ceiling at a height  $h$  above the floor. The speed of projection is  $\sqrt{(8gh)}$  and the direction of projection makes an acute angle  $\alpha$  with the horizontal. The particle strikes the ceiling and rebounds, the impact being perfectly elastic. Show that for this to happen  $\alpha$  must be at least  $\frac{1}{6}\pi$  and that the range on the floor is then

$$8h \cos \alpha (2 \sin \alpha - \sqrt{(4 \sin^2 \alpha - 1)}).$$

Another particle  $P_2$  is projected from  $O$  with the same velocity as  $P_1$  but its impact with the ceiling is perfectly inelastic. Find the difference  $D$  between the ranges of  $P_1$  and  $P_2$  on the floor and show that, as  $\alpha$  varies,  $D$  has a maximum value when  $\alpha = \frac{1}{4}\pi$ .

14



The end  $O$  of a smooth light rod  $OA$  of length  $2a$  is a fixed point. The rod  $OA$  makes a fixed angle  $\sin^{-1}(\frac{3}{5})$  with the downward vertical  $ON$ , but is free to rotate about  $ON$ . A particle of mass  $m$  is attached to the rod at  $A$  and a small ring  $B$  of mass  $m$  is free to slide on the rod but is joined to  $A$  by a spring of natural length  $a$  and modulus of elasticity  $kmg$ . The vertical plane containing the rod  $OA$  rotates about  $ON$  with constant angular velocity  $\sqrt{5g/(2a)}$  and  $B$  is at rest relative to the rod. Show that the length of  $OB$  is

$$\frac{(10k + 8)a}{10k - 9}.$$

Given that the reaction of the rod on the particle at  $A$  makes an angle  $\tan^{-1}(\frac{13}{21})$  with the horizontal, find the value of  $k$ . Find also the magnitude of the reaction between the rod and the ring  $B$ .

15 A pack of  $2n$  (where  $n \geq 4$ ) cards consists of two each of  $n$  different sorts. If four cards are drawn from the pack without replacement show that the probability that no pairs of identical cards have been drawn is

$$\frac{4(n-2)(n-3)}{(2n-1)(2n-3)}.$$

Find the probability that exactly one pair of identical cards is included in the four.

If  $k$  cards are drawn without replacement and  $2 < k < 2n$ , find an expression for the probability that there are exactly  $r$  pairs of identical cards included when  $r < \frac{1}{2}k$ .

For even values of  $k$  show that the probability that the drawn cards consist of  $\frac{1}{2}k$  pairs is

$$\frac{1 \times 3 \times 5 \times \dots \times (k-1)}{(2n-1)(2n-3)\dots(2n-k+1)}.$$

16 The random variables  $X$  and  $Y$  take integer values  $x$  and  $y$  respectively which are restricted by  $x \geq 1$ ,  $y \geq 1$  and  $2x + y \leq 2a$  where  $a$  is an integer greater than 1. The joint probability is given by

$$P(X = x, Y = y) = c(2x + y),$$

where  $c$  is a positive constant, within this region and zero elsewhere. Obtain, in terms of  $x$ ,  $c$  and  $a$ , the marginal probability  $P(X = x)$  and show that

$$c = \frac{6}{a(a-1)(8a+5)}.$$

Show that when  $y$  is an even number the marginal probability  $P(Y = y)$  is

$$\frac{3(2a-y)(2a+2+y)}{2a(a-1)(8a+5)}$$

and find the corresponding expression when  $y$  is odd.

Evaluate  $E(Y)$  in terms of  $a$ .