THE ROYAL STATISTICAL SOCIETY

2001 EXAMINATIONS – SOLUTIONS

HIGHER CERTIFICATE

PAPER I – STATISTICAL THEORY

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Higher Certificate, Paper I, 2001. Question 1

(i) A and B are independent. So $P(\overline{A} \cap \overline{B}) = P(\overline{A})P(\overline{B})$.

$P(\overline{A}) = 1 - P(A) = \frac{1}{3}$; $P(\overline{B}) = \frac{1}{2}$; so $P(\overline{A} \cap \overline{B}) = \frac{1}{6}$.

(ii) $\overline{A} \cap \overline{B} = [(\overline{A} \cap \overline{B}) \cap C] \cup [(\overline{A} \cap \overline{B}) \cap \overline{C}]$,

with the two events $[(\overline{A} \cap \overline{B}) \cap C]$ and $[(\overline{A} \cap \overline{B}) \cap \overline{C}]$ being disjoint.

Hence $P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A} \cap \overline{B}) - P(\overline{A} \cap \overline{B} \cap C) = \frac{1}{6} - \frac{1}{10} = \frac{1}{15}$.

(iii) $B \cap C = (A \cap B \cap C) \cup (\overline{A} \cap B \cap C)$, these being disjoint.

Further, $(\overline{A} \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) = (\overline{A} \cap B)$.

Hence
$$\begin{aligned} P(B \cap C) &= P(A \cap B \cap C) + P(\overline{A} \cap B) - P(\overline{A} \cap B \cap \overline{C}) \\ &= \frac{1}{4} + \left(\frac{1}{3} \times \frac{1}{2}\right) - \left[P(\overline{A} \cap \overline{C}) - P(\overline{A} \cap \overline{B} \cap \overline{C})\right] \end{aligned}$$
since $\overline{A}, B$ are independent.

But A, C are also independent and so are $\overline{A}, \overline{C}$.

Therefore $P(B \cap C) = \frac{1}{4} + \frac{1}{6} - \left(\frac{1}{3} \times \frac{2}{5}\right) + \frac{1}{15} = \frac{5}{12} - \frac{1}{15} = \frac{63}{12 \times 15} = \frac{7}{20}$.

(iv) $P(B \mid C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{7/20}{3/5} = \dfrac{7}{12}$.

(v) $P(A \mid B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)} = \dfrac{1/4}{7/20} = \dfrac{5}{7}$.

(vi) $P(A \cap B \mid A \cap C) = \dfrac{P(A \cap B \cap C)}{P(A \cap C)} = \dfrac{\frac{1}{4}}{\frac{2}{3} \times \frac{3}{5}} = \dfrac{5}{8}$

(A, C are independent).

<u>Higher Certificate, Paper I, 2001. Question 2</u>

(a) Fix the position of M_1 (suppose him to be the host).

	M_1 (1)		Label the positions clockwise.
(6)		(2)	M_2, M_3, W_1, W_2, W_3 may be arranged in $5! = 120$ ways.
(5)		(3)	
	(4)		

(i) M_2, M_3 must occupy (3) and (5); W_1, W_2, W_3 may occupy the other places in 3! ways, making $2\times 3!$ arrangements. The probability is then $\frac{12}{120}=\frac{1}{10}$.

(ii) M_2, M_3 must occupy (2) and (6) <u>or</u> (2) and (3) <u>or</u> (5) and (6). In each case, M_2, M_3 can be placed in two orders, making 6 positions altogether for the three men. The women may again fill the remaining places in 3! ways.

The probability is $\frac{6\times 6}{120}=\frac{3}{10}$.

(iii) EITHER $1-\frac{1}{10}-\frac{3}{10}=\frac{3}{5}$, because this is the only other arrangement possible besides (i) and (ii);

OR by having M_2 in (2), M_3 in (4) or (5); M_2 in (6), M_3 in (3) or (4); M_2 in (3), M_3 in (4); M_2 in (4), M_3 in (5); or any of these with M_2, M_3 interchanged, giving 12 positionings of the men. There are again 3! orders for the women, so the probability is $\frac{6\times 12}{120}=\frac{3}{5}$.

(b) (i) Event D is "has disease", T is "tests positive".

$$P(D\,|\,T)=\frac{P(D\cap T)}{P(T)}=\frac{P(T\,|\,D)P(D)}{P(T)}=\frac{P(T\,|\,D)P(D)}{P(T\,|\,D)P(D)+P(T\,|\,\bar{D})P(\bar{D})}$$
$$=\frac{p_1p_0}{p_1p_0+(1-p_2)(1-p_0)}.$$

(ii) $$\frac{0.95\times 0.005}{(0.95\times 0.005)+(0.05\times 0.995)}=\frac{0.00475}{0.0545}=0.0872.$$

The error rates in the clinical tests are large compared to the chance of having the disease, so the calculated probability is very small.

Higher Certificate, Paper I, 2001. Question 3

(a) (i) $P(S \geq 2300) = 1 - P(S < 2300) = 1 - \Phi\left(\dfrac{2300-2000}{300}\right)$

$= 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

$P(H \geq 2300) = 1 - \Phi\left(\dfrac{2300-2500}{125}\right) = 1 - \Phi(-1.6)$

$= 1 - 0.0548 = 0.9452$.

(ii) $P(S > H) = P(S - H > 0)$, where $(S - H)$ is N(–500, 90000+15625) i.e. N(–500, 325^2). Hence $P(S > H) = \Phi\left(\dfrac{-500}{325}\right) = \Phi(-1.5385) = 0.0620$.

(b) (i) The lifetime X is S with probability 0.6 and H with probability 0.4.

Hence $E[X] = 0.6 \times 2000 + 0.4 \times 2500 = 2200$ hrs.

$$P(X > 2600) = P(X > 2600 \mid S)P(S) + P(X > 2600 \mid H)P(H)$$

$$= \Phi\left(-\frac{600}{300}\right) \times 0.6 + \Phi\left(-\frac{100}{125}\right) \times 0.4$$

(using the appropriate tail areas from Normal tables)

$$= 0.6\,\Phi(-2) + 0.4\,\Phi(-0.8) = 0.6 \times 0.02275 + 0.4 \times 0.2119$$

$$= 0.01365 + 0.08476 = 0.09841.$$

(ii) $P(H \mid \; > 2600) = \dfrac{P(X > 2600 \mid H)P(H)}{P(X > 2600)} = \dfrac{0.08476}{0.09841} = 0.8613$.

(iii) $\bar{X}$ will have mean 2200. It is not Normally distributed but we way apply the Central Limit Theorem if we know its variance. In large samples we may take $\bar{X}$ as approximately N(2200, 346.8^2), so that

$$P(\bar{X} > 2300) = 1 - \Phi\left(\frac{2300-2200}{346.8/\sqrt{100}}\right) = 1 - \Phi\left(\frac{100}{34.68}\right) = \Phi(-2.8835) = 0.002.$$

<u>Higher Certificate, Paper I, 2001. Question 4</u>

An easy method is to consider X as $\sum X_i$, where X_i are a set of n Bernoulli variables with $P(X_i = 1) = p, P(X_i = 0) = (1 - p)$. Then $E[X_i] = p,$ so $E[X] = np$.

Also $E\left[X_i^2\right] = p,$ so $\operatorname{Var}(X_i) = p - p^2$ and $\operatorname{Var}(X) = n\left(p - p^2\right) = npq.$

ALTERNATIVELY: $$E[X] = \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x} = \sum_{x=1}^{n} \frac{n!\,p^x q^{n-x}}{(x-1)!(n-x)!}$$

$$= np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{(n-1)-(x-1)} = np.$$

Similarly, $\operatorname{Var}(\mathrm{X}) = E\left[X(X-1)\right] + E[X] - \left(E[X]\right)^2$, and we have

$$E\left[X(X-1)\right] = \sum_{x=0}^{n} x(x-1)\binom{n}{x}p^x q^{n-x} = \sum_{x=2}^{n} x(x-1)\binom{n}{x}p^x q^{n-x}$$

$$= n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}q^{(n-2)-(x-2)} = n(n-1)p^2,$$

and hence $\operatorname{Var}(\mathrm{X}) = n(n-1)p^2 + np - n^2p^2 = np - np^2 = npq$.

PGFs or MGFs could also be used.

(i) (a) $0.75^{16} \approx 0.0100226 = 0.0100$ approx.

(b) $1 - P$(no one gets all 16 right), probability is $1 - \left\{1 - 0.75^{16}\right\}^{12}$
$= 1 - \{0.9899774\}^{12} = 0.1139$.

(ii) $P(B - A > 0)$ can be studied using a Normal approximation to the difference $B - A$, i.e. $\mathrm{N}\left(16\{0.5 - 0.75\}, 16\{(0.5\times0.5) + (0.75\times0.25)\}\right)$, i.e. $\mathrm{N}(-4, 7)$.

The probability is found as $P\left(B - A > \frac{1}{2}\right)$ using a continuity correction since $B - A$ takes discrete values.

Hence it is $1-\Phi\left(\dfrac{0.5-(-4)}{\sqrt{7}}\right)=\Phi\left(-\dfrac{4.5}{\sqrt{7}}\right)=\Phi(-1.7008)\approx 0.0445$.

[Note: this would be 0.0653 without the continuity correction.]

(iii) $E\left[\bar{X}\right]=E\left[X\right]=np=16\times 0.75=12$ in set A.

Similarly, $E\left[\bar{Y}\right]=16\times 0.5=8$ in set B.

There are 12 students in A and 25 in B, so that

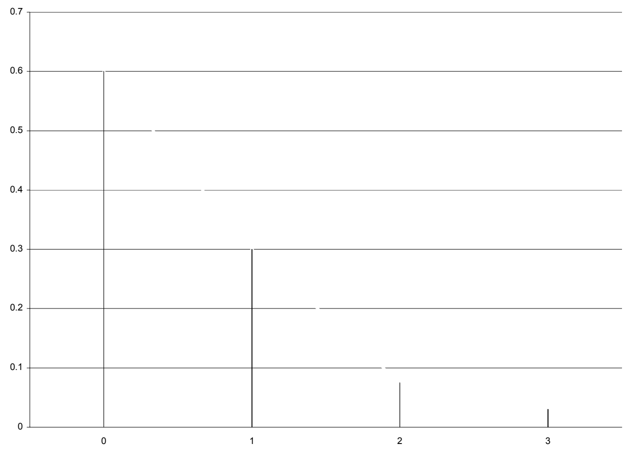
$$\operatorname{Var}\left(\bar{X}\right)=\frac{16\times 0.75\times 0.25}{12}=\frac{1}{4} \quad \text{in set } A$$

$$\operatorname{Var}\left(\bar{Y}\right)=\frac{16\times 0.5\times 0.5}{25}=\frac{4}{25} \quad \text{in set } B.$$

Higher Certificate, Paper I, 2001. Question 5

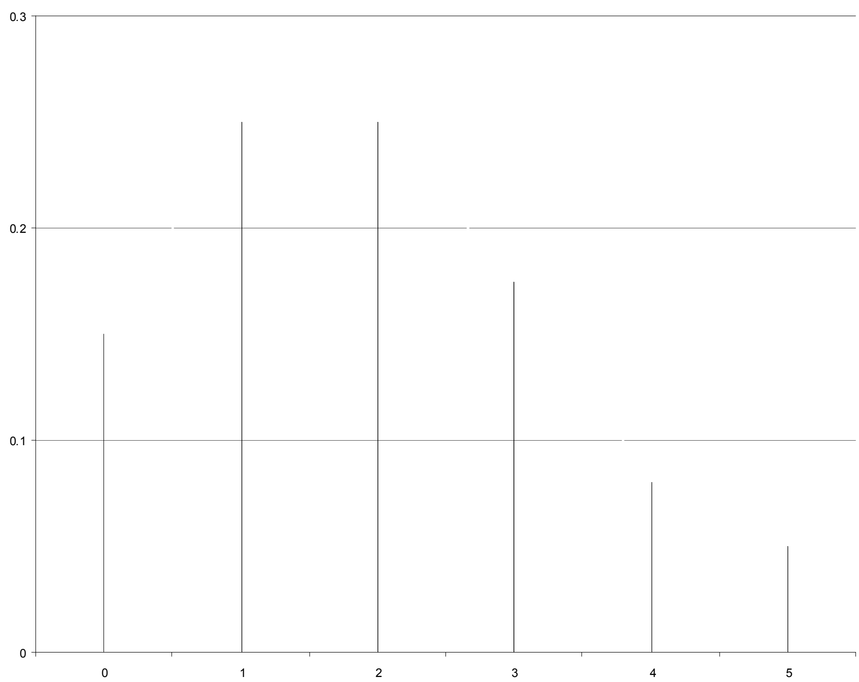
$$\frac{p(x+1)}{p(x)} = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}.\frac{x!}{e^{-\lambda}\lambda^{x}} = \frac{\lambda}{x+1} \quad \text{for } x = 0, 1, 2, \dots .$$

For $\lambda = \frac{1}{2}$, $p(0) = 0.60653$; so $p(1) = 0.30327$, $p(2) = 0.07582$, $p(3) = 0.01264$.



Graph of $p(x)$ for λ = ½.

For $\lambda = 2$, $p(0) = 0.13534$;
so $p(1) = 0.27067 = p(2)$, $p(3) = 0.18045$, $p(4) = 0.09022$, $p(5) = 0.03609$.



Graph of $p(x)$ for λ = 2.

$$M_X(t) = E\left[e^{Xt}\right] = \sum_{x=0}^{\infty} \frac{e^{xt}e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda}.e^{\lambda e^t} = \exp\left(\lambda\left\{e^t - 1\right\}\right).$$

$\dfrac{\partial M}{\partial t} = \lambda e^t e^{\lambda\left(e^t-1\right)}$; put $t = 0$ and this is λ, which is therefore $E[X]$.

$\dfrac{\partial^2 M}{\partial t^2} = \left(\lambda^2 e^{2t} + \lambda e^t\right)\left(e^{\lambda\left(e^t-1\right)}\right)$; put $t = 0$ and this is $\lambda^2+\lambda$, but it is also $E\left[X^2\right]$.

Hence $\quad \mathrm{Var}(\mathrm{X}) = E\left[X^2\right] - \left(E[X]\right)^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda$.

$\dfrac{\partial^3 M}{\partial t^3} = \left(\lambda^3 e^{3t} + 3\lambda^2 e^{2t} + \lambda e^t\right)\left(e^{\lambda\left(e^t-1\right)}\right) = \lambda^3 + 3\lambda^2 + \lambda$ at $t = 0$. This is $E\left[X^3\right]$.

Now, $E\left[(X-\lambda)^3\right] = E\left[X^3\right] - 3\lambda E\left[X^2\right] + 3\lambda^2 E[X] - \lambda^3$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda\left(\lambda^2 + \lambda\right) + 3\lambda^2.\lambda - \lambda^3 = \lambda.$$

For $Y = X_1 + X_2 + \ldots + X_n$ we have

$$E\left[e^{Yt}\right] = \prod_{i=1}^{n} \mathrm{E}\left[e^{X_i t}\right] = \left(M_X(t)\right)^n = \exp\left[\lambda n\left(e^t - 1\right)\right].$$

This is the mgf of a Poisson distribution with parameter λn and so $E[Y]$ = Var(Y) = λn.

$$P(Y \geq 40) = 1 - P(Y \leq 39) \approx 1 - \Phi\left(\frac{39.5-25}{5}\right),$$

using continuity correction, and $\mu = \lambda n = 25$. This is $1 - \Phi(2.9) = 0.00187$.

With a positively skew distribution, the Normal approximation is likely to underestimate the probability in the right hand tail and so we expect this answer to be <u>less</u> than the true value.

Higher Certificate, Paper I, 2001. Question 6

$$L(p)=\prod_{i=1}^{n}\left(q^{x_i-1}p\right)=p^n q^{\sum x_i-n}=\left(\frac{p}{1-p}\right)^n(1-p)^{n\bar{x}}$$

$$\ln L=n\ln p-n\ln(1-p)+n\bar{x}\ln(1-p)$$

$\dfrac{\partial}{\partial p}(\ln L)=\dfrac{n}{p}+\dfrac{n}{1-p}-\dfrac{n\bar{x}}{1-p}=0$ when $\dfrac{1}{\hat{p}}=\dfrac{-1+\bar{x}}{1-\hat{p}}$ which gives $\hat{p}=\dfrac{1}{\bar{x}}$ as m.l.estimate.

$\dfrac{\partial^2(\ln L)}{\partial p^2}=-\dfrac{n}{p^2}-\dfrac{n(\bar{x}-1)}{(1-p)^2}$ which is < 0, confirming the maximum.

$$E\left[\frac{\partial^2\ln L}{\partial p^2}\right]=\frac{n}{p^2}+\frac{n}{(1-p)^2}\left[E(\bar{X})-1\right]=\frac{n}{p^2}+\frac{n}{(1-p)^2}\left(\frac{1}{p}-1\right)=\frac{n}{p^2}+\frac{n}{p(1-p)}$$

$$=\frac{n}{p^2(1-p)}$$

Hence $\operatorname{Var}(\hat{p})\approx\dfrac{p^2(1-p)}{n}$.

$\sum fx=448,\ \sum f=56,\ \hat{p}=\dfrac{56}{448}=0.125$.

$\operatorname{Var}(\hat{p})=\dfrac{0.125^2\times0.875}{56}=0.0002441,\quad \mathrm{SE}(\hat{p})=0.015625$.

Approximate 95% confidence interval for p is $\hat{p}\pm1.96\,\mathrm{SE}(\hat{p})$, which is

$0.125\pm1.96\times0.015625$, i.e. 0.125 ± 0.030625 or (0.0944, 0.1556).

When $p=\dfrac{1}{6}$, we have $\dfrac{1}{\bar{X}}\sim \mathrm{N}\left(\dfrac{1}{6},\dfrac{5}{216\times56}\right)$, i.e. N(0.1667, 0.00041336); therefore

the probability of obtaining $\hat{p}\le0.125$ is approximately

$$\Phi\left(\frac{0.125-0.1667}{0.020331}\right)=\Phi(-2.0496)=-0.0202.$$

The confidence interval for p did not include $\dfrac{1}{6}$; also now the probability being very small is consistent with rejecting a null hypothesis that $p=\dfrac{1}{6}$, i.e. that the die is fair.

<u>Higher Certificate, Paper I, 2001. Question 7</u>

(a) $P(X_i \le x) = F(x)$ for $i = 1, 2, ..., n$.

(i) $$F_{X_{\max}}(x) = P(X_1 \le x,\ X_2 \le x,\ ...,\ X_n \le x) = \prod_{i=1}^{n} P(X_i \le x)$$
$$= \left[F(x)\right]^n$$

(ii) For $X_{\min} \ge x$, we require <u>every</u> X_i to be $\ge x$.

Now, $P(X_i \le x) = F(x)$, so $P(X_i \ge x) = 1 - F(x)$, for all i.

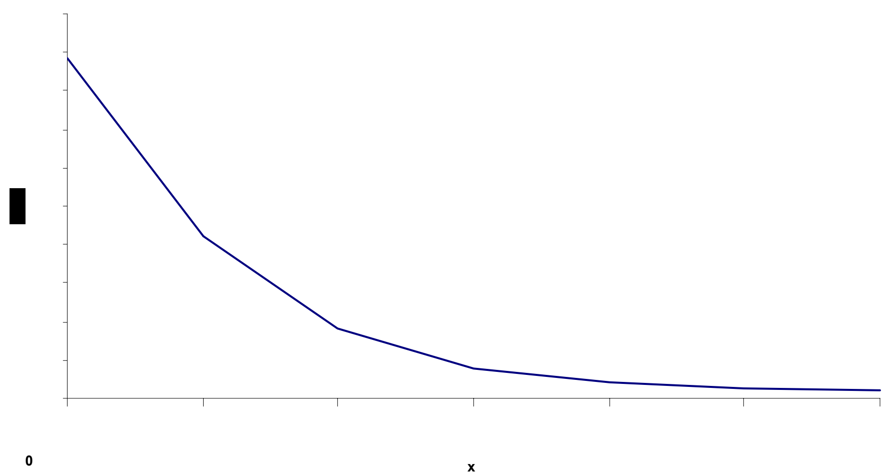
$$\therefore F_{X_{\min}}(x) = P(X_{\min} \le x) = 1 - P(X_{\min} \ge x) = 1 - P(X_1 \ge x, ..., X_n \ge x)$$
$$= 1 - \left[1 - F(x)\right]^n$$

(iii) Pdfs are derivatives of cdfs:

$$f_{X_{\max}} = n\left[F(x)\right]^{n-1} f(x)$$

$$f_{X_{\min}} = n\left[1 - F(x)\right]^{n-1} f(x), \qquad \text{since } \frac{\partial F(x)}{\partial x} = f(x).$$

(b)



$$F(x) = \int_0^x \frac{\alpha}{(1+u)^{\alpha+1}}\,du = \left[-\frac{1}{(1+u)^{\alpha}} \right]_0^x = 1 - \frac{1}{(1+x)^{\alpha}}.$$

Median M is such that $F(M) = \frac{1}{2}$. So we have

$$1 - \frac{1}{(1+M)^{\alpha}} = \frac{1}{2} \quad \text{or} \quad \frac{1}{(1+M)^{\alpha}} = \frac{1}{2} \quad \text{or } 2 = (1+M)^{\alpha} \quad \text{or} \quad M = 2^{(1/\alpha)} - 1.$$

Using (a)(ii), $F_{X_{\min}} = 1 - \left(\frac{1}{(1+x)^{\alpha}} \right)^n = 1 - \frac{1}{(1+x)^{n\alpha}}$, also Pareto but with α replaced by $n\alpha$.

The median of $X_{\min}$ is then $2^{\frac{1}{n\alpha}} - 1$, which is $2^{1/n} - 1$ if $\alpha = 1$. We require $2^{1/n} - 1 < 0.1$ or $2^{1/n} < 1.1$, i.e. $\frac{1}{n} \ln 2 < \ln 1.1$, giving $n > \frac{\ln 2}{\ln 1.1} = \frac{0.6931}{0.0953} = 7.27$. Hence $n \geq 8$.

<u>Higher Certificate, Paper I, 2001. Question 8</u>

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

where y is the response (observation) and x the value of the explanatory variable on that unit; $\{\varepsilon_i\}$ is a set of independent, identically distributed random variables with mean 0 and the same variance σ^2. Usually they are assumed Normal as a basis for inference. x_i is assumed "fixed", not "random".

(i) (a) There is an increasing trend, and the relationship between y and x appears curvilinear.

(b) In simple regression R^2 is the square of the correlation, r, between x and y. In general it is the proportion of the variance of y which can be explained by the dependence of y on all explanatory variables $\{x_i\}$ in the model; hence it is the square of the correlation between $\hat{y}$ and y.

In the ANOVA , $\dfrac{\text{regression SS}}{\text{total SS}} = \dfrac{269.33}{335.37} = 0.803, \text{ or } 80.3\%$.

(c) As x (% operating capacity) increases by 1 unit so y (profit) increases by 0.31562 units.

A 95% confidence interval is $0.31562 \pm t_{10} \times 0.04942$, which is 0.31562 ± 0.11011 or $(0.2055, 0.4257)$.

(d) Values of profit have been predicted for capacity 25%, 50% and 75%. The confidence intervals for these predictions are those given; but note that 25% is far outside the range of available data (hence the remark about extreme x values). A 95% confidence interval is an interval which should cover the <u>true</u> y at a given x with probability 0.95, based on the fitted linear regression.

(ii) (a) The logarithmic plot shows that a linear regression in these units is a much better fit. There is still an increasing trend.

(b) $\log_{10}(\text{profits}) = -0.519 + 0.0177(\text{capacity})$

i.e. $\text{profits} = 10^{-0.519+0.0177(\text{capacity})}$.

(c) For capacity = 25, the 95% limits are –0.2731 and +0.1210, and the actual prediction is –0.0760, in $\log_{10}$ units. Anti-logging these (i.e. raising 10 to these powers) we find the 95% limits are 0.5332 and 1.3213.

The 'prediction' is 0.8395.

These limits do not overlap the limits on the previous model.

The prediction now is for a small profit, compared with a loss on the previous model.

(iii) The scatter plots indicate that the logarithmic model is preferred, and so do the plots of residuals which show a random pattern (as compared with a systematic, curved one for the previous model). R^2 also higher (92.4%) on the log model.

But a log model cannot predict negative profits – i.e. losses – which are quite possible in general though not for these data if used within the range of x values given.

Extrapolation down to 25% is well outside the data and so is not reliable on any model.