THE ROYAL STATISTICAL SOCIETY

2005 EXAMINATIONS – SOLUTIONS

GRADUATE DIPLOMA

STATISTICAL THEORY AND METHODS

PAPER I

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(i) If $\{E_1, E_2, \ldots, E_n\}$ partition S, then $P(A) = \sum_{i=1}^{n} P(A \mid E_i)P(E_i)$. This is the law of total probability.

Since $P(A \cap E_j) = P(A | E_j)P(E_j) = P(E_j | A)P(A)$, we have (Bayes' Theorem)

$$P(E_j \mid A) = \frac{P(A \mid E_j)P(E_j)}{P(A)} = \frac{P(A \mid E_j)P(E_j)}{\sum_{i=1}^{n} P(A \mid E_i)P(E_i)}.$$

(ii) Let Y be the amount of time spent in the fitting room; Y is exponential with parameter $\frac{1}{3x}$.

(a) $P(Y < y \mid x) = \int_{t=0}^{y} \frac{1}{3x} e^{-t/3x} dt = \left[-e^{-t/3x}\right]_{t=0}^{y} = 1 - e^{-y/3x}$ (for $y > 0$).

Since x takes the values 1, 2, 3, 4 each with probability ¼, we therefore have

$$F(y) = \left(1 - e^{-y/3} + 1 - e^{-y/6} + 1 - e^{-y/9} + 1 - e^{-y/12}\right) \times \frac{1}{4} \quad \text{(using (i))}.$$

When $y = 5$ this is $1 - \frac{1}{4}\left(e^{-5/3} + e^{-5/6} + e^{-5/9} + e^{-5/12}\right)$

$$= 1 - \frac{1}{4}(0.1889 + 0.4346 + 0.5738 + 0.6592) = 1 - 0.464$$

and so $P(Y > 5) = 0.464$.

Solution continued on next page

(b) Let X be the number of garments taken to the room. Then

$$E(X) = \frac{1}{4}(1+2+3+4) = \frac{5}{2},$$

$$E(X^2) = \frac{1}{4}(1+4+9+16) = \frac{15}{2},$$

so $\text{Var}(X) = \dfrac{15}{2} - \dfrac{25}{4} = \dfrac{5}{4}$.

[These results may be quoted, as X has a discrete uniform distribution.]

Now, $E(Y|X) = 3X$. Also, because Y has an exponential distribution,

$$\text{Var}(Y|X) = (3X)^2 = 9X^2.$$

Thus

$$E(Y) = E\{E(Y|X)\} = E\{3X\} = 3E(X) = \frac{15}{2}.$$

Also,

$$\begin{aligned}
\text{Var}(Y) &= E\{\text{Var}(Y|X)\} + \text{Var}\{E(Y|X)\} \\
&= E\{9X^2\} + \text{Var}\{3X\} \\
&= 9E(X^2) + 9\text{Var}(X) \\
&= 9 \times \frac{15}{2} + 9 \times \frac{5}{4} \\
&= \frac{315}{4}.
\end{aligned}$$

<u>Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 2</u>

(i) $X + Y$ can take the values 0, 1, 2,…, $n + m$. For these values,

$$P(X+Y=z)=\sum_{x=0}^{z}P(X=x \text{ and } Y=z-x)$$

$$=\sum_{x=0}^{z}P(X=x)P(Y=z-x)$$

$$=\sum_{x=0}^{z}\left\{\binom{n}{x}\theta^{x}(1-\theta)^{n-x}\binom{m}{z-x}\theta^{z-x}(1-\theta)^{m-z+x}\right\}$$

$$=\theta^{z}(1-\theta)^{m+n-z}\sum_{x=0}^{z}\binom{n}{x}\binom{m}{z-x}$$

$$=\binom{m+n}{z}\theta^{z}(1-\theta)^{m+n-z}.$$

Thus $X + Y$ has the binomial distribution with parameters $m + n$ and θ.

[An alternative method is to use probability generating functions.]

(ii) $P\left(X=x \middle| X+Y=z\right)$

$$=\frac{P(X=x\cap X+Y=z)}{P(X+Y=z)}$$

$$=\frac{P(X=x\cap Y=z-x)}{P(X+Y=z)}$$

$$=\frac{\binom{n}{x}\theta^{x}(1-\theta)^{n-x}\binom{m}{z-x}\theta^{z-x}(1-\theta)^{m-z+x}}{\binom{m+n}{z}\theta^{z}(1-\theta)^{m+n-z}}$$

$$=\frac{\binom{n}{x}\binom{m}{z-x}}{\binom{m+n}{z}}$$

(i.e. a hypergeometric distribution).

Solution continued on next page

(iii) Let X and Y be the numbers of failed components in the two networks. We have $n = 20$, $m = 30$, $\theta = 0.1$, $z = 6$ in the above notation.

$$P(X = 3 \mid X + Y = 6) = \frac{\binom{20}{3}\binom{30}{3}}{\binom{50}{6}} = \frac{20.19.18.30.29.28.6.5.4.3.2.1}{3.2.1.3.2.1.50.49.48.47.46.45} = 0.2913.$$

$$f(x,y) = 12x^2 \qquad (0 < x < y < 1)$$

(i)

$$E\left(X^r Y^s\right) = \int_{x=0}^{1}\int_{y=x}^{1} 12x^{2+r} y^s \, dy \, dx$$

$$= 12\int_0^1 x^{2+r}\left[\frac{y^{s+1}}{s+1}\right]_{y=x}^{1} dx$$

$$= \frac{12}{s+1}\int_0^1 x^{2+r}(1 - x^{s+1})\, dx$$

$$= \frac{12}{s+1}\left[\frac{x^{r+3}}{r+3} - \frac{x^{r+s+4}}{r+s+4}\right]_0^1$$

$$= \frac{12}{s+1}\left(\frac{1}{r+3} - \frac{1}{r+s+4}\right)$$

$$= \frac{12(s+1)}{(s+1)(r+3)(r+s+4)} = \frac{12}{(r+3)(r+s+4)}.$$

Hence

$$E(X) = \frac{12}{4\times 5} = \frac{3}{5} \qquad \text{(put } r = 1,\ s = 0\text{; similarly for the others)}$$

$$E(Y) = \frac{12}{3\times 5} = \frac{4}{5}$$

$$E\left(X^2\right) = \frac{12}{5\times 6} = \frac{2}{5}, \quad \text{so } \operatorname{Var}(X) = \frac{2}{5} - \left(\frac{3}{5}\right)^2 = \frac{1}{25}$$

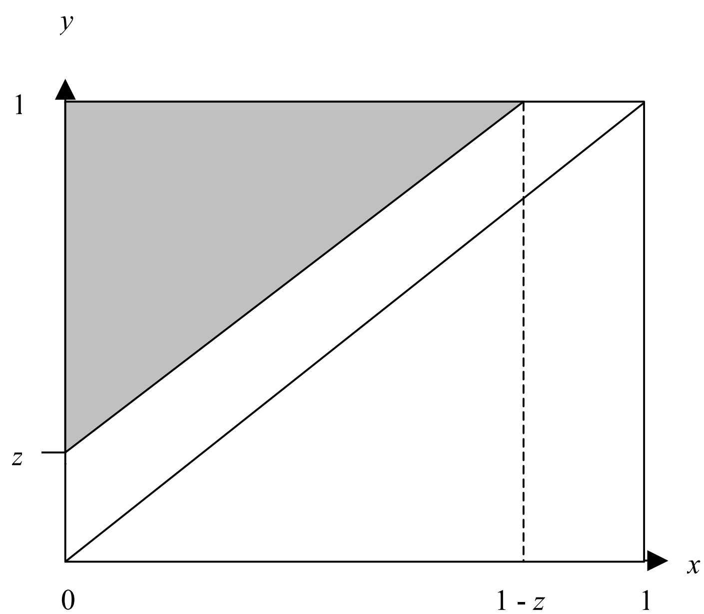
$$E\left(Y^2\right) = \frac{12}{3\times 6} = \frac{2}{3}, \quad \text{so } \operatorname{Var}(Y) = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

$$E(XY) = \frac{12}{4\times 6} = \frac{1}{2}, \quad \text{so } \operatorname{Cov}(X,Y) = \frac{1}{2} - \left(\frac{3}{5}\times\frac{4}{5}\right) = \frac{1}{50}$$

$$\therefore \rho_{XY} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{\frac{1}{50}}{\sqrt{\frac{1}{25}\times\frac{2}{75}}} = \frac{\sqrt{3}}{2\sqrt{2}}$$

Solution continued on next page

(ii) $P(Y-X>z) = \int_{x=0}^{1-z}\left\{\int_{y=x+z}^{1} 12x^2 dy\right\} dx$, the evaluation being over the shaded region shown:



This is $12\int_0^{1-z} x^2 \left[y\right]_{x+z}^{1} dx = 12\int_0^{1-z} x^2(1-x-z) \quad dx$

$$= 12\int_0^{1-z}\left\{x^2(1-z)-x^3\right\} dx = 12\left[(1-z)\frac{x^3}{3}-\frac{x^4}{4}\right]_0^{1-z}$$

$$= 12\left\{\frac{(1-z)^4}{3}-\frac{(1-z)^4}{4}\right\} = (1-z)^4$$

Therefore $F(z) = 1-(1-z)^4 \qquad (\text{for } 0 \le z \le 1)$

and $f(z) = F'(z) = 4(1-z)^3 \qquad (\text{for } 0 \le z \le 1)$.

Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 4

(i) $U = \dfrac{X}{X+Y}$, $V = X + Y$. So $X = UV$ and $Y = (1 - U)V$.

The Jacobian of the transformation is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v(1-u) + uv = v\,.$$

The joint pdf of X, Y is $f(x,y) = \dfrac{\theta^{\alpha+\beta} x^{\alpha-1} y^{\beta-1} e^{-\theta(x+y)}}{\Gamma(\alpha)\Gamma(\beta)}$, for $x > 0,\ y > 0$.

Hence the joint pdf of U, V is

$$g(u,v) = f(x,y)|J| = \frac{\theta^{\alpha+\beta}(uv)^{\alpha-1}(1-u)^{\beta-1}v^{\beta-1}e^{-\theta v}v}{\Gamma(\alpha)\Gamma(\beta)} \qquad (\text{for } v > 0,\ 0 < u < 1)$$

$$= \frac{\theta^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}\left\{u^{\alpha-1}(1-u)^{\beta-1}\right\}\left\{v^{\alpha+\beta-1}e^{-\theta v}\right\}.$$

This is of the form of a product

constant × function of u alone [$g(u)$, say] × function of v alone [$h(v)$, say]

and so U, V are independent. $g(u)$ is proportional to $u^{\alpha-1}(1 - u)^{\beta-1}$, the pdf of a beta distribution, and so U has a beta distribution. $h(v)$ is proportional to $v^{\alpha+\beta-1}e^{-\theta v}$, the pdf of a gamma distribution, and so V has a gamma distribution. The scale parameter of V is θ, as for X and Y.

(ii) $U = \dfrac{X}{X+Y}$ is the required distribution, where X and Y are the common exponential random variables. Taking $\alpha = \beta = 1$, $g(u) = u^0(1 - u)^0 = 1$ and so U has the uniform distribution on (0, 1).

(i) $M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} \frac{1}{\sqrt{2\pi x}} e^{-x/2} dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-1/2} e^{-x\left(\frac{1}{2}-t\right)} dx$.

$t < ½$ is used in what follows to ensure convergence of the integral.

Write $u = x\left(\frac{1}{2}-t\right)$, so that $du = \left(\frac{1}{2}-t\right) dx$.

Then $M_X(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left(\frac{u}{\frac{1}{2}-t}\right)^{-1/2} e^{-u} \frac{1}{\frac{1}{2}-t} du$

$$= \frac{2}{\sqrt{2\pi}\sqrt{2}\sqrt{1-2t}} \int_0^\infty u^{-1/2} e^{-u} du$$

The integral here should be recognised as $\Gamma(½) = \sqrt{\pi}$; alternatively, refer back to the original pdf

$$= \frac{1}{\sqrt{1-2t}}.$$

$M_X(t) = (1-2t)^{-1/2}$, so $M'_X(t) = -\frac{1}{2}(1-2t)^{-3/2}(-2) = (1-2t)^{-3/2}$

$\therefore E(X) = M'_X(0) = 1$

$M''_X(t) = -\frac{3}{2}(1-2t)^{-5/2}(-2) = 3(1-2t)^{-5/2}$

$\therefore E\left(X^2\right) = M''_X(0) = 3$

$\therefore \text{Var}(X) = E\left(X^2\right) - \{E(X)\}^2 = 3 - 1^2 = 2$.

Solution continued on next page

(ii) $M_{X_1+\ldots+X_n}(t)=\{M_X(t)\}^n$

$$=(1-2t)^{-n/2}.$$

Now using $M_{aY+b}(t)=e^{bt}M_Y(at)$,

$$M_z(t)=e^{-\sqrt{\frac{n}{2}}\,t}\left(1-2\frac{t}{\sqrt{2n}}\right)^{-n/2}.$$

To find the limiting form of $M_Z(t)$, we take logs:

$$\log M_Z(t)=-\sqrt{\frac{n}{2}}\,t\;-\;\frac{n}{2}\log\left(1-t\sqrt{\frac{2}{n}}\right)$$

$$=-t\sqrt{\frac{n}{2}}\;-\;\frac{n}{2}\left\{-t\sqrt{\frac{2}{n}}-\frac{1}{2}\left(t\sqrt{\frac{2}{n}}\right)^2-\frac{1}{3}\left(t\sqrt{\frac{2}{n}}\right)^3-\ldots\right\}$$

$$=\frac{1}{2}t^2\;+\;\frac{1}{3}t^3\sqrt{\frac{2}{n}}\;+\;\ldots$$

$$\to\frac{t^2}{2}\quad\text{as } n\to\infty.$$

So $M_Z(t)\to e^{t^2/2}$ as $n\to\infty$, which is the mgf of N(0, 1).

Therefore in the limit *Z* becomes N(0, 1), i.e. the standard Normal distribution.

<u>Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 6</u>

(i) In a $U(-\theta, \theta)$ distribution, $f(x) = \dfrac{1}{2\theta}$ and $F(x) = \dfrac{x}{2\theta} + \dfrac{1}{2}$, for $-\theta < x < \theta$.

$$F(u_{(1)}, u_{(n)}) = P(U_{(n)} \le u_{(n)}) - P(U_{(1)} > u_{(1)} \text{ and } U_{(n)} \le u_{(n)})$$

$$= P(\text{all data} \le u_{(n)}) - P(\text{all data between } u_{(1)} \text{ and } u_{(n)})$$

$$= \{F(u_{(n)})\}^n - \{F(u_{(n)}) - F(u_{(1)})\}^n$$

$$= \left\{\frac{u_{(n)}}{2\theta}\right\}^n - \left\{\frac{u_{(n)} - u_{(1)}}{2\theta}\right\}^n, \quad \text{for } -\theta < u_{(1)} < u_{(n)} < \theta.$$

$$f(u_{(1)}, u_{(n)}) = \frac{\partial^2}{\partial u_{(1)} \partial u_{(n)}} F(u_{(1)}, u_{(n)})$$

$$= \frac{n(n-1)(u_{(n)} - u_{(1)})^{n-2}}{(2\theta)^n}.$$

[An argument using the multinomial distribution with one observation at each of $u_{(1)}$ and $u_{(n)}$ and with $n - 2$ in between is also acceptable.]

(ii) Transforming to $R = U_{(n)} - U_{(1)}$ and $T = U_{(1)}$ (so that $U_{(n)} = R + T$), we have the Jacobian

$$J = \frac{\partial\left(u_{(1)}, u_{(n)}\right)}{\partial(r, t)} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \quad \text{so } |J| = 1.$$

Hence $f(r, t) = \dfrac{n(n-1)r^{n-2}}{(2\theta)^n}$ (for $-\theta < t < \theta - r$, $0 < r < 2\theta$).

$$\therefore f(r) = \int_{-\theta}^{\theta - r} \frac{n(n-1)r^{n-2}}{(2\theta)^n} dt$$

$$= \frac{n(n-1)r^{n-2}(2\theta - r)}{(2\theta)^n}, \quad \text{for } 0 < r < 2\theta.$$

Solution continued on next page

(iii)

$$
\begin{aligned}
E(R) &= \frac{n(n-1)}{(2\theta)^n}\int_0^{2\theta} r^{n-1}(2\theta - r)\,dr \\
&= \frac{n(n-1)}{(2\theta)^n}\int_0^{2\theta} (2\theta r^{n-1} - r^n)\,dr \\
&= \frac{n(n-1)}{(2\theta)^n}\left[\frac{2\theta r^n}{n} - \frac{r^{n+1}}{n+1}\right]_0^{2\theta} \\
&= \frac{n(n-1)}{(2\theta)^n}\times\frac{(2\theta)^{n+1}}{n(n+1)} \\
&= 2\theta\left(\frac{n-1}{n+1}\right).
\end{aligned}
$$

Hence $\frac{1}{2}R$ is a biased estimator of θ (but asymptotically unbiased as $n \to \infty$).

Graduate Diploma, Statistical Theory & Methods, Paper I, 2005. Question 7

(i) (a) The inverse cumulative distribution function method can be used with tables of the standard Normal cdf $\Phi(x)$. The values of z are such that $\Phi(z) = u$, and for the four values of u the corresponding values of z are $-1.07, -0.42, +0.46, +1.40$.

(b) These can be transformed to N(–2, 0.81) by $w = \mu + \sigma z$ or $w = -2 + 0.9z$, to give $-2.963, -2.378, -1.586, -0.740$.

(c) The chi-squared distribution with one degree of freedom is the square of N(0, 1), so take values of z^2 from (i): 1.14, 0.18, 0.21, 1.96.

(ii) The probabilities and cumulative probabilities for a Poisson distribution with mean 2 are:

r	0	1	2	3	4	5	…
$P(r)$	0.1353	0.2707	0.2707	0.1804	0.0902	0.0361	
$F(r)$	0.1353	0.4060	0.6767	0.8571	0.9473	0.9834	

Taxis: 0.553 corresponds to $r = 2$ (it is between 0.4060 and 0.6767) etc, giving 2, 3, 1, 5, 1.

Similarly for customers: 3, 1, 1, 2, 2.

Time	*Taxis*	*Arrivals*	*Customers*	*Arrivals*
3.00	0	2	0	3
3.01	0	3	1	1
3.02	1	1	0	1
3.03	1	5	0	2
3.04	4	1	0	2
3.05	3		0	

(i) (a)

$$\text{If } \mathbf{C} = \begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix} \text{ then } \mathbf{C}^{-1} = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{CDC}^{-1} = \frac{1}{\alpha+\beta}\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\alpha+\beta}\begin{bmatrix} 1 & \alpha(\alpha+\beta-1) \\ 1 & \beta(1-\alpha-\beta) \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\alpha+\beta}\begin{bmatrix} \alpha+\beta-\alpha^2-\alpha\beta & \alpha^2+\alpha\beta \\ \alpha\beta+\beta^2 & \alpha+\beta-\alpha\beta-\beta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$= \mathbf{P}$$

(b) The n-step transition matrix is $\mathbf{P}^n$, which can be written $\left(\mathbf{CDC}^{-1}\right)\left(\mathbf{CDC}^{-1}\right)\left(\mathbf{CDC}^{-1}\right)\cdots\left(\mathbf{CDC}^{-1}\right)$ and every pair $\mathbf{C}^{-1}\mathbf{C}$ is replaced by $\mathbf{I}$ to give $\mathbf{CD}^n\mathbf{C}^{-1}$.

$\mathbf{D}^n$ is simply $\begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix}$, i.e. $\begin{bmatrix} 1 & 0 \\ 0 & \lambda^n \end{bmatrix}$ in the given notation.

Since $0 < \alpha < 1$ and $0 < \beta < 1$, we have $-1 < \lambda < 1$, i.e. $|\lambda| < 1$; therefore $\lambda^n \to 0$.

Thus $\mathbf{P}^n \to \mathbf{C}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{C}^{-1}$ which is

$$\begin{bmatrix} 1 & -\alpha \\ 1 & \beta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} \times \frac{1}{\alpha+\beta} = \frac{1}{\alpha+\beta}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ -1 & 1 \end{bmatrix} = \frac{1}{\alpha+\beta}\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix}.$$

Solution continued on next page

(ii) Let state 0 be no rain and state 1 be rain. The transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{bmatrix}.$$

This is the matrix in (i) with $\alpha = 0.2$ and $\beta = 0.9$.

As there is no rain on the first visit, $\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{P}^n$ gives the probabilities for the two states on the next visit in n days' time. As n is large, $\mathbf{P}^n$ can be taken as approximately equal to the limiting value in (i)(b), i.e. here

$$\tfrac{1}{1.1}\begin{bmatrix} 0.9 & 0.2 \\ 0.9 & 0.2 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{P}^n = \begin{bmatrix} \frac{0.9}{1.1} & \frac{0.2}{1.1} \end{bmatrix},$$

i.e. $P(\text{rain}) = \frac{0.2}{1.1} = \frac{2}{11}$.

Replacing $\begin{bmatrix} 1 & 0 \end{bmatrix}$ with $\begin{bmatrix} 0 & 1 \end{bmatrix}$ for the first visit gives the same answer because of the form of $\mathbf{P}^n$. In the long run there are about 9 days without rain for every 2 days with rain.