2017 PAT Courses

Dates:	30 Sep - 1 Oct 2017 14-15 Oct 2017 21-22 Oct 2017
Location:	Imperial College London

1. A jar contains buttons of four different colours. There are twice as many yellow as green, twice as many red as yellow and twice as many blue as red. What is the probability of taking from the jar:

   a) a blue button

   b) a red button

   c) a yellow button

   d) a green button

   (You may assume that you are only taking one button at a time and replacing it in the jar before selecting the next colour.) [4]

Y	G	R	B	Tot
2	1	4	8	15

a) $\frac{8}{15}$ b) $\frac{4}{15}$ c) $\frac{2}{15}$ d) $\frac{1}{15}$

2. What is the sum of the following terms:

$$1 + e^{-x} + e^{-2x} + \dots$$

Over what range of x is the solution valid? [4]

G.P. with $a = 1$, $r = e^{-x}$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-e^{-x}}$$

Valid for $|r| < 1$

$$|e^{-x}| < 1$$

$$e^{-2x} < 1$$

$$-2x < \ln 1$$

$$\underline{x > 0}$$

3. Evaluate the integrals:

a) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x} \mathrm{d}x$

b) $\displaystyle\int_0^{2} \frac{x}{x^2+6x+8} \mathrm{d}x.$

[6]

a) $$\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,dx = \Big[\ln|1+\sin x|\Big]_0^{\pi/2}$$
$$= \ln 2 - \ln 1$$
$$= \underline{\ln 2}$$

b) $$\int_0^2 \frac{x}{x^2+6x+8}\,dx = \int_0^2 \left(\frac{-4/-2}{x+4} + \frac{-2/2}{x+2}\right)dx$$
$$= 2\int_0^2 \frac{1}{x+4}\,dx - \int_0^2 \frac{1}{x+2}\,dx$$
$$= \Big[2\ln|x+4| - \ln|x+2|\Big]_0^2$$
$$= \left[\ln\left|\frac{(x+4)^2}{x+2}\right|\right]_0^2 = \ln 9 - \ln 8$$
$$= \underline{\ln\left(\frac{9}{8}\right)}$$

4. What is the coefficient of x^7 in the expansion of $(1+2x)^4(1-2x)^6$? [4]

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \end{array}$$

$(1)\ 1^4(2x)^0$

$(4)\ 1^3(2x)^1$ *

$(6)\ 1^2(2x)^2$ □

$(4)\ 1^1(2x)^3$ △

$(1)\ 1^0(2x)^4$ ○

$(1)\ 1^6(-2x)^0$

$(6)\ 1^5(-2x)^1$

$(15)\ 1^4(-2x)^2$

$(20)\ 1^3(-2x)^3$ ○

$(15)\ 1^2(-2x)^4$ △

$(6)\ 1^1(-2x)^5$ □

$(1)\ 1^0(-2x)^6$ *

$$\text{Coeff} = 4(2)(-2)^6 + 36(2)^2(-2)^5 + 60(2)^3(-2)^4 + 20(2)^4(-2)^3$$

$$= 4(128) - 36(128) + 60(128) - 20(128)$$

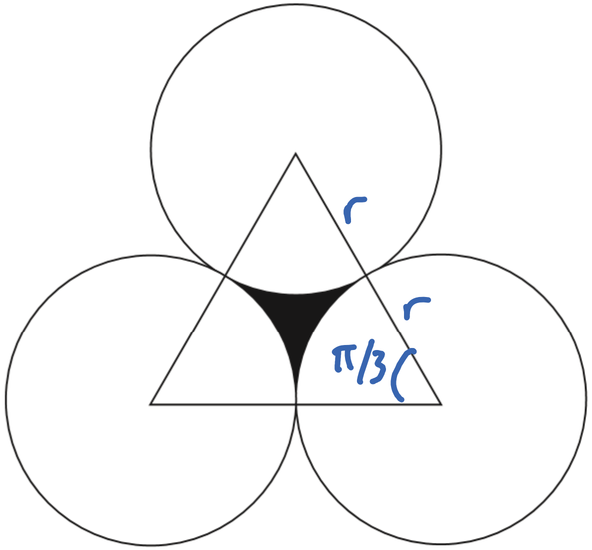
$$= 128(4 - 36 + 60 - 20)$$

$$= 128 \times 8$$

$$= \underline{1024}$$

5. Consider the shape shown below. What is the area of the equilateral triangle (with length of side $= 2r$) which is not enclosed within the circles (each with radius $= r$), and which is shown shaded black in the figure?

[5]



$$A = A_{triangle} - 3A_{sector}$$

$$A_{sector} = \frac{\pi}{3} \times \frac{1}{2} \times r^2 = \frac{\pi r^2}{6}$$

$$A_{triangle} = \frac{1}{2}(2r)^2 \times \sin\frac{\pi}{3}$$

$$= \frac{1}{2} \cdot 4r^2 \cdot \frac{\sqrt{3}}{2}$$

$$= \sqrt{3}\, r^2$$

$$\Rightarrow A = \sqrt{3}\, r^2 - 3 \times \frac{\pi r^2}{6}$$

$$= \left(\sqrt{3} - \frac{\pi}{2}\right) r^2$$

6. You want to make a snowman out of modelling clay. The snowman consists of 2 spheres, where one sphere has a radius r, the other has a radius $2r$. The modelling clay comes in the form of a cylinder with radius $r/2$. What length of modelling clay is required to make the snowman? [5]

$$V_{snow} = \frac{4}{3}\pi r^3 + \frac{4}{3}\pi (2r)^3$$

$$= 12\pi r^3$$

$$V_{clay} = \pi \left(\frac{r}{2}\right)^2 \cdot \ell = 12\pi r^3$$

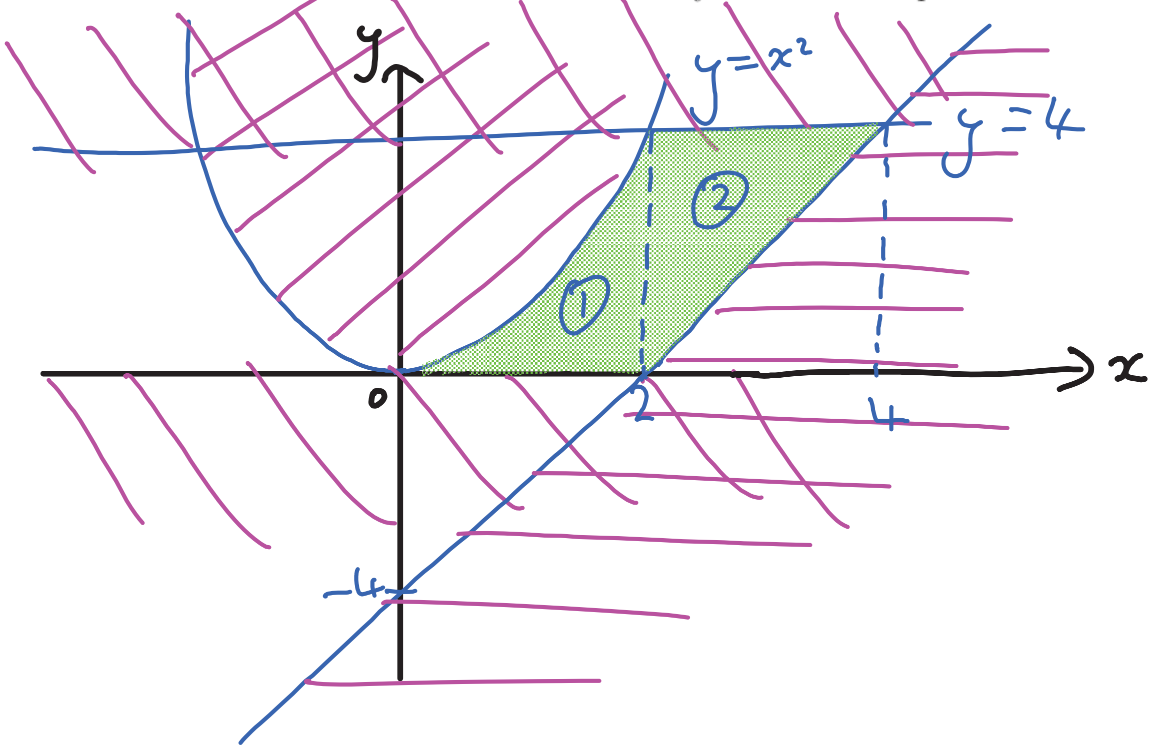
$$\ell = \frac{12\pi r^3}{\pi r^2/4}$$

$$= \underline{48r}$$

7. Sketch the region defined by:

$y \le x^2$ and $4 \ge y \ge 0$ and $y \ge 2x - 4$.

Evaluate the area defined by the above inequalities. [7]



$4 = 2x - 4$

$x = 4$

$$1: \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

$$2: \frac{4 \times 2}{2} = 4$$

$$\Rightarrow A = 4 + \frac{8}{3} = \frac{20}{3}$$

8. If $f(x) = e^x$ for $x < 0$ and $f(x) = e^{-x} + 2x$ for $x \geq 0$, sketch the function $f(x)$ and its first, second and third derivatives. [8]

	$x < 0$	$x \geqslant 0$
$f(x)$	e^{x}	$e^{x}+2x$
$f'(x)$	e^{x}	$-e^{-x}+2$
$f''(x)$	e^{x}	e^{-x}
$f'''(x)$	e^{x}	$-e^{-x}$

$f'(x) = 0$

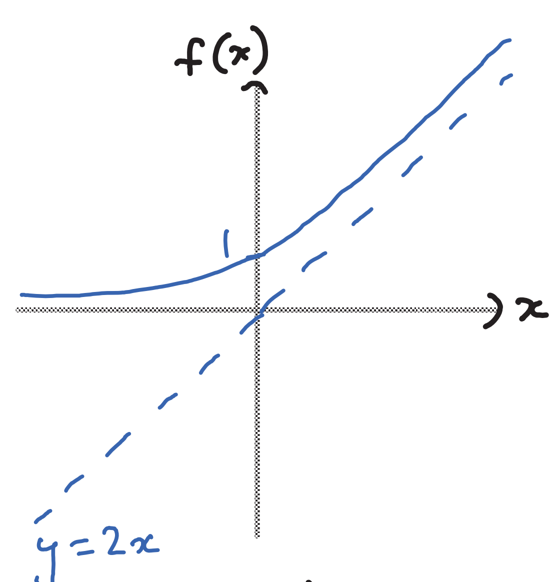
$-e^{-x}+2=0$

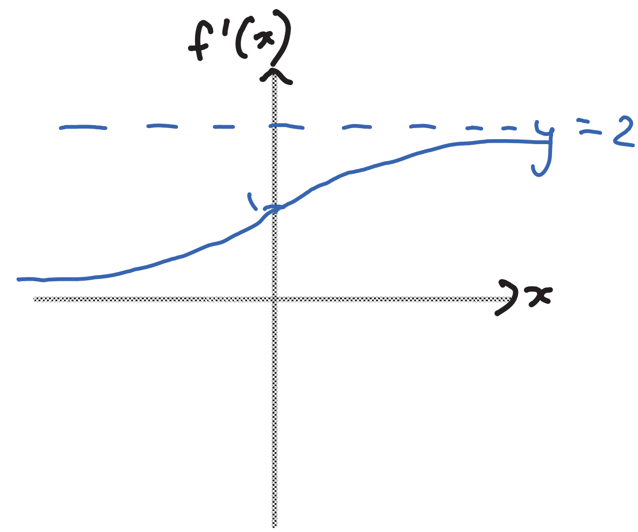
$e^{-x}=2$

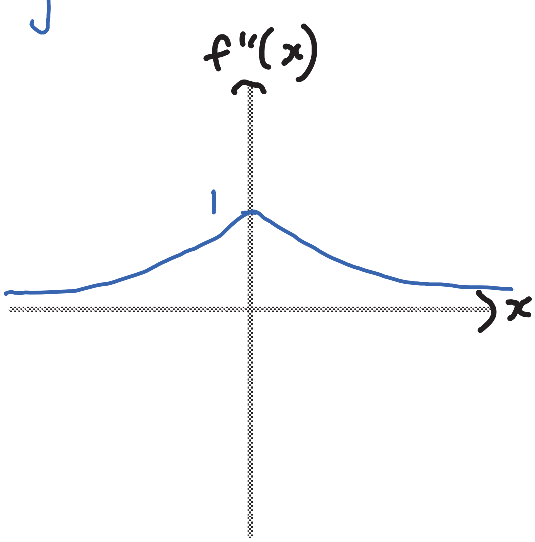
$-x = \ln 2$

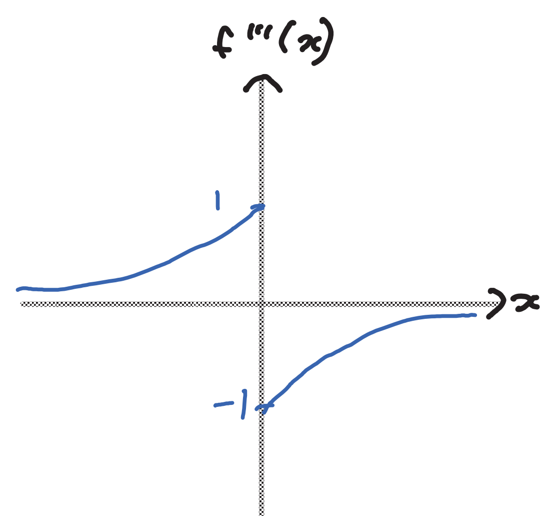
$x = \ln\frac{1}{2}$

$\Rightarrow$ min pt. -ve

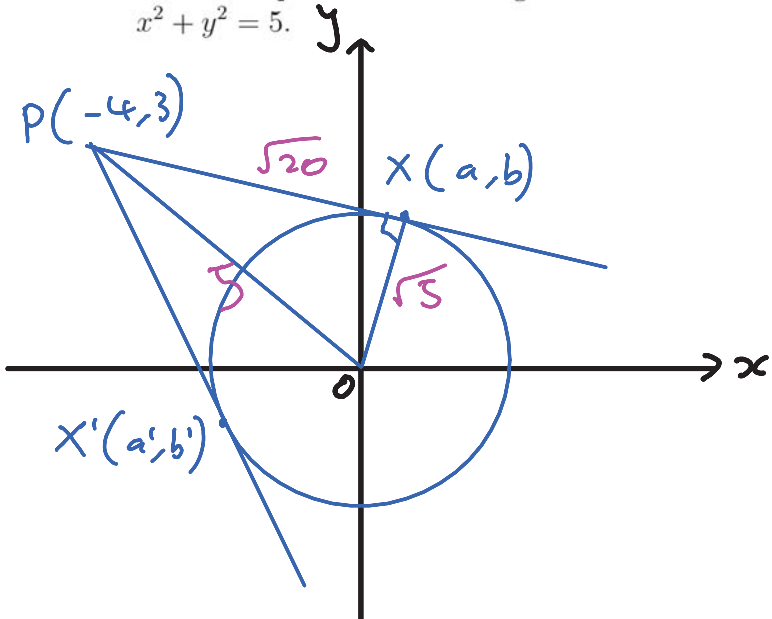








9. Find the equations of the tangents drawn from the point (-4, 3) to the circle $x^2 + y^2 = 5$. [7]



$OP^2 = 3^2 + (-4)^2 = 25$

$PX^2 = OP^2 - OX^2$

$= 25 - 5$

$= 20$

Find X:

$PX: \quad (a+4)^2 + (b-3)^2 = 20$

$a^2 + 8a + 16 + b^2 - 6b + 9 = 20$

$a^2 + b^2 + 8a - 6b = -5 \quad (1)$

$OX: \quad a^2 + b^2 = 5 \quad (2)$

$(1)-(2): \quad 8a - 6b = -10$

$b = \dfrac{4a+5}{3} \quad (3)$

$(1): \quad a^2 + \left(\dfrac{4a+5}{3}\right)^2 + 8a - 6\left(\dfrac{4a+5}{3}\right) = -5$

$9a^2 + 16a^2 + 40a + 25 + 72a - 72a - 90 = -5$

$25a^2 + 40a - 20 = 0$

$5a^2 + 8a - 4 = 0$

$(5a - 2)(a+2) = 0$

$a = \frac{2}{5} \quad \text{or} \quad a' = -2$

$(3): \quad b = \dfrac{4(2/5) + 5}{3} = \dfrac{11}{5}$

$b' = \dfrac{4(-2) + 5}{3} = -1$

Find eqn's:

$PX: m = \dfrac{11/5 - 3}{2/5 + 4} = \dfrac{11-15}{2+20}$

$= -\frac{2}{11}$

$y - 3 = -\frac{2}{11}(x+4)$

$2x + 11y - 25 = 0$

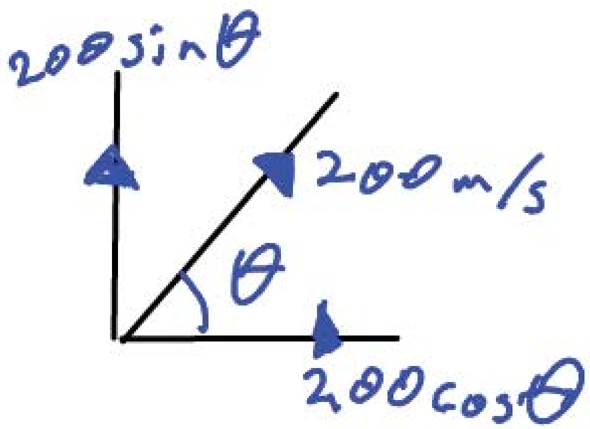
$PX': \quad m = \dfrac{-1-3}{-2+4} = -2$

$y - 3 = -2(x+4)$

$2x + y + 5 = 0$

10 A gun is designed that can launch a projectile, of mass 10 kg, at a speed of 200 m/s. The gun is placed close to a straight, horizontal railway line and aligned such the projectile will land further down the line. A small rail car, of mass 200 kg and travelling at a speed of 100 m/s passes the gun just as it is fired. Assuming the gun and the car are at the same level, at what angle upwards must the projectile be fired in order that it lands in the rail car?

[3]



$$200\cos\theta = 100$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

11 An electron gun in a cathode ray tube accelerates an electron with mass m and charge $-e$ across a potential difference of 50 V and directs it horizontally towards a fluorescent screen 0.4 m away. How far does the electron fall during its journey to the screen? Take $m \approx 10^{-30}$ kg and $e \approx 1.6 \times 10^{-19}$ C.

[5]

$$\frac{1}{2}mv^2 = eV$$

$$v^2 = \frac{2 \times 1.6 \times 10^{-19} \times 50}{10^{-30}} = 16 \times 10^{12}$$

$$v = 4 \times 10^{6}\ \text{ms}^{-1}$$

Time to screen, $t = \dfrac{0.4}{4 \times 10^{6}} = 1 \times 10^{-7}\ \text{s}$

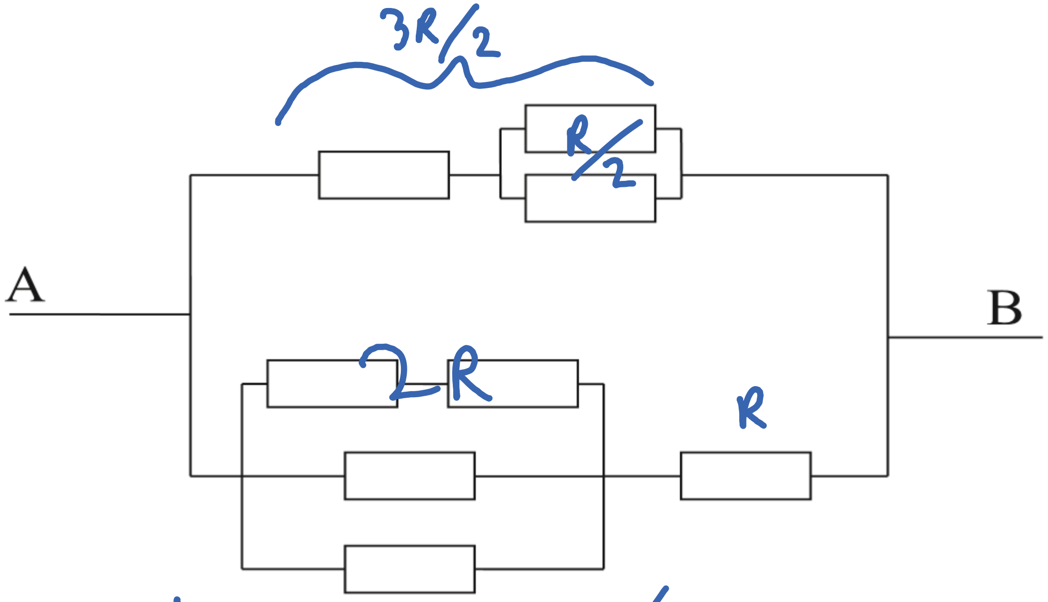
$\downarrow$ +ve $s = ?,\ u = 0,\ v = x,\ a = 10\ \text{ms}^{-2},\ t = 1 \times 10^{-7}\ \text{s}$

$$s = ut + \frac{1}{2}at^2$$

$$= \frac{1}{2} \times 10 \times \left(1 \times 10^{-7}\right)^2$$

$$= 5 \times 10^{-14}\ \text{m}$$

12 Given the circuit below, where all resistors have the same value ($= R$), what is the resistance between A and B? [5]



$$\frac{1}{\frac{1}{R} + \frac{1}{R} + \frac{1}{2R}} = \frac{2R}{5}$$

$$\frac{7R}{5}$$

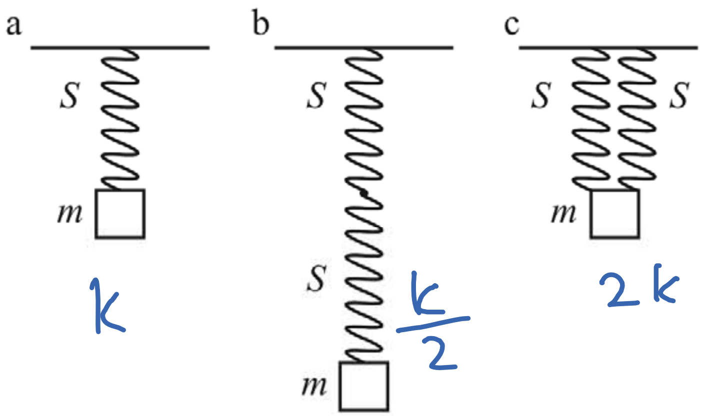
$$R_T = \frac{1}{\frac{2}{3R} + \frac{5}{7R}}$$

$$= \frac{1}{\frac{14+15}{21R}}$$

$$= \underline{\frac{21R}{29}}$$

13 A mass m is attached to a spring S (as sketched in figure **a** below) and oscillates with a period T. What would be the period of the oscillation if two springs S are connected in series (figure **b**) or in parallel (figure **c**)?

What would be the period of the oscillations in case (a) on a planet with surface gravity $2g$? [4]



$$T = 2\pi\sqrt{\frac{m}{k}}$$

series: $T' = 2\pi\sqrt{\frac{m}{k/2}} = \sqrt{2}\,T$

parallel: $T' = 2\pi\sqrt{\frac{m}{2k}} = \frac{T}{\sqrt{2}}$

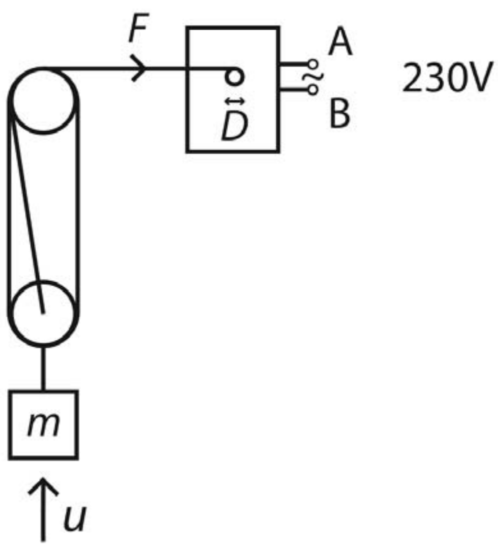
other planet: same because period doesn't depend on g

14 An electric motor is lifting a mass via a system of pulleys as sketched below. The motor is powered by a voltage source of 230 V. The diameter of the motor winding reel is D = 5 cm and a mass m = 100 kg is being lifted with a speed u = 0.5 m/s. The masses of the pulleys and the string can be neglected.

a) What is the electric current driving the motor?

b) What is the angular velocity of the motor's winding reel?

c) What is the force F with which the motor is pulling? [5]



a) $P = IV$

$P = \frac{mgh}{t} = \frac{100 \times 10 \times 0.5}{1}$

$= 500W$

$\Rightarrow I = \frac{500}{230} = \frac{50}{23} A$

b) $v = \omega r$

$v = 3u = 1.5 ms^{-1}$ $r = D/2 = 0.025 m$

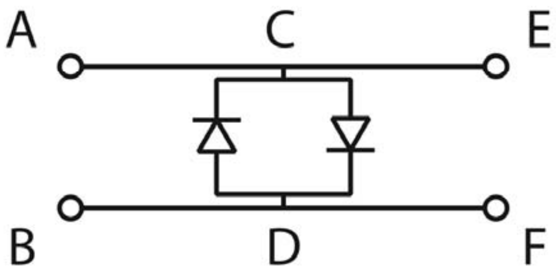
$\omega = \frac{1.5}{0.025} = \frac{15}{0.25} = 60 rads^{-1}$

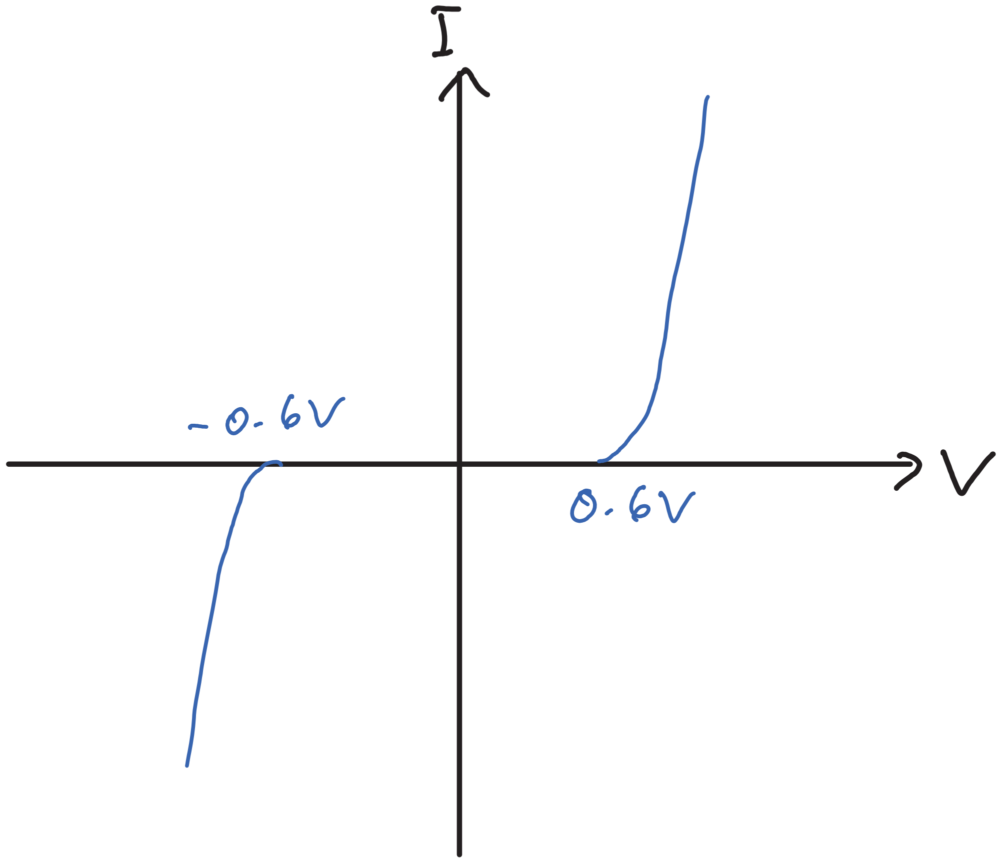
c) $P = Fv$

$F = \frac{P}{v} = \frac{500}{1.5} = 333 N$

15 Two diodes are connected as sketched below. Sketch the current flowing between points C and D as a function of voltage applied between points C and D.

A sensitive amplifier is connected to terminals E and F to measure small electric signals from an instrument connected to terminals A and B. From time to time there are discharges in the instrument which might destroy the amplifier if the amplifier is connected to the instrument directly, without the diodes. Explain briefly how the diodes protect the amplifier. [4]





For small signals (low voltage), current doesn't flow through C-D. Amplifier operates.

When there's a discharge (high voltage), current flows through C-D, not through the amplifier

16 A catapult consists of a massless cup attached to a massless spring of length l and of spring constant k. If a ball of mass m is loaded into the cup and the catapult pulled back to extend the spring to a total length x, what velocity does the ball reach when launched horizontally? [2]

The catapult is then used to launch the ball vertically. If the spring is extended to the same total length of x before release, to what velocity does the catapult now accelerate the ball? [2]

i) $KE = EPE$

$$\frac{1}{2}mv^2 = \frac{1}{2}k(\Delta x)^2$$

$$v = \sqrt{\frac{k}{m}}\,(x-l)$$

ii) $$\frac{1}{2}mv_1^2 + mg(x-l) = \frac{1}{2}k(x-l)^2$$

$$v_1^2 = \frac{k}{m}(x-l)^2 - 2g(x-l)$$

$$v_1 = \sqrt{\frac{k}{m}(x-l)^2 - 2g(x-l)}$$

17 For this question you may assume that the electrostatic potential energy of two positively charged particles (with charge $+Q_1$ and $+Q_2$) separated by a distance x is given by

$$k\frac{Q_1 Q_2}{x}$$

where k is a constant.

Two charged particles are placed a distance d apart from each other. One has charge $= +Q$ and mass $= m$, whilst the other has charge $= +2Q$ and mass $= 2m$. The charges are initially held stationary, but are then released. Find an expression for the maximum speed of the particle with mass $= 2m$.

[6]

Particle 1: $2m$, $2Q$, v

Particle 2: m, Q, v'

CoM: $0 = 2mv - mv'$

$v' = 2v$

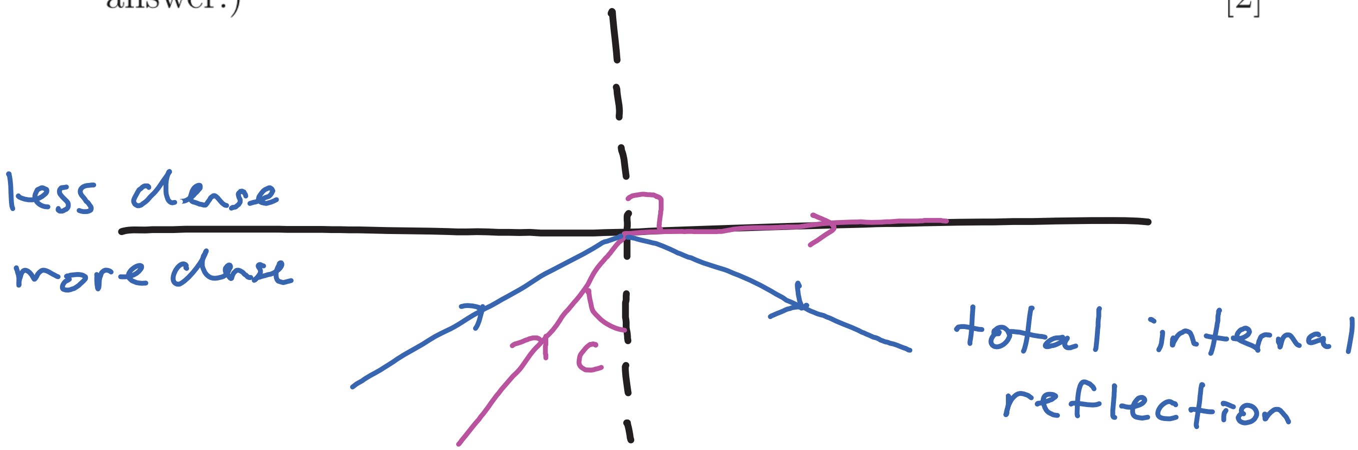
CoE: $\frac{1}{2}m(v')^2 + \frac{1}{2}(2m)v^2 = \frac{2kQ^2}{d}$

$\frac{1}{2}m4v^2 + mv^2 = \frac{2kQ^2}{d}$

$v = \sqrt{\frac{2kQ^2}{3md}}$

18 This question concerns total internal reflection, optical fibres and refraction. You may assume that the refractive index of glass is larger than that of water, and that the refractive index of water is larger than that of air.

a) Explain what is meant by the phrases "total internal reflection" and "critical angle". (You are encouraged to use a diagram to explain your answer.) [2]



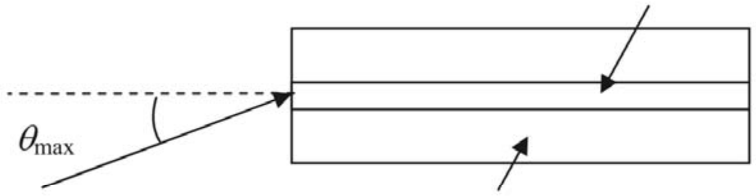
b) Derive an equation relating the critical angle and the refractive indices of two materials, n_1 and n_2 where $n_2 < n_1$. [2]

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

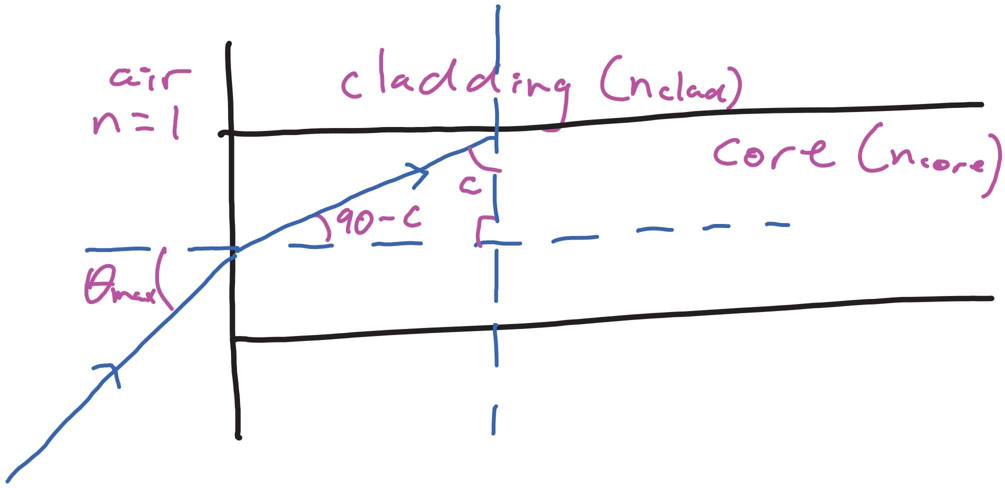
$$n_1 \sin c = n_2 \sin 90$$

$$\underline{\sin c = \frac{n_2}{n_1}}$$

c) An optical fibre is usually made of two materials - a core and a cladding as shown in the diagram below (not drawn to scale).



Light may only be transmitted along the fibre if the incident angle of the light is less than a maximum angle θ_{max}. By using your expression from part b) and Snell's law, or otherwise, derive an expression for θ_{max} in terms of the core and cladding refractive indices only. [3]



part b: $\sin c = \frac{n_2}{n_1} = \frac{n_{core}}{n_{clad}}$

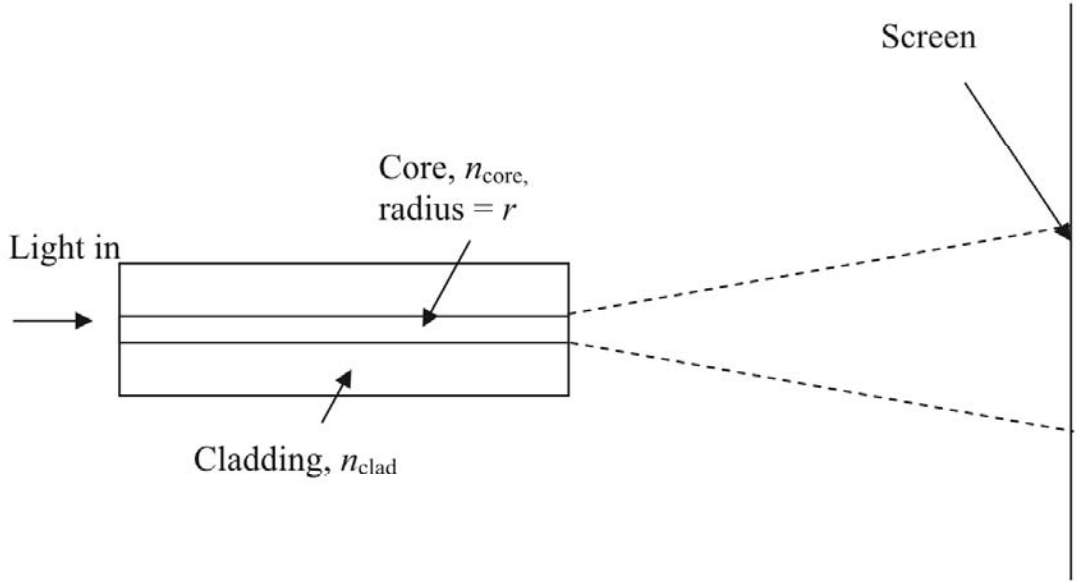
Snell's law: $\sin\theta_{max} = n_{core}\sin(90-c)$

$= n_{core}\cos c$

$= n_{core}\sqrt{1-\sin^2 c}$

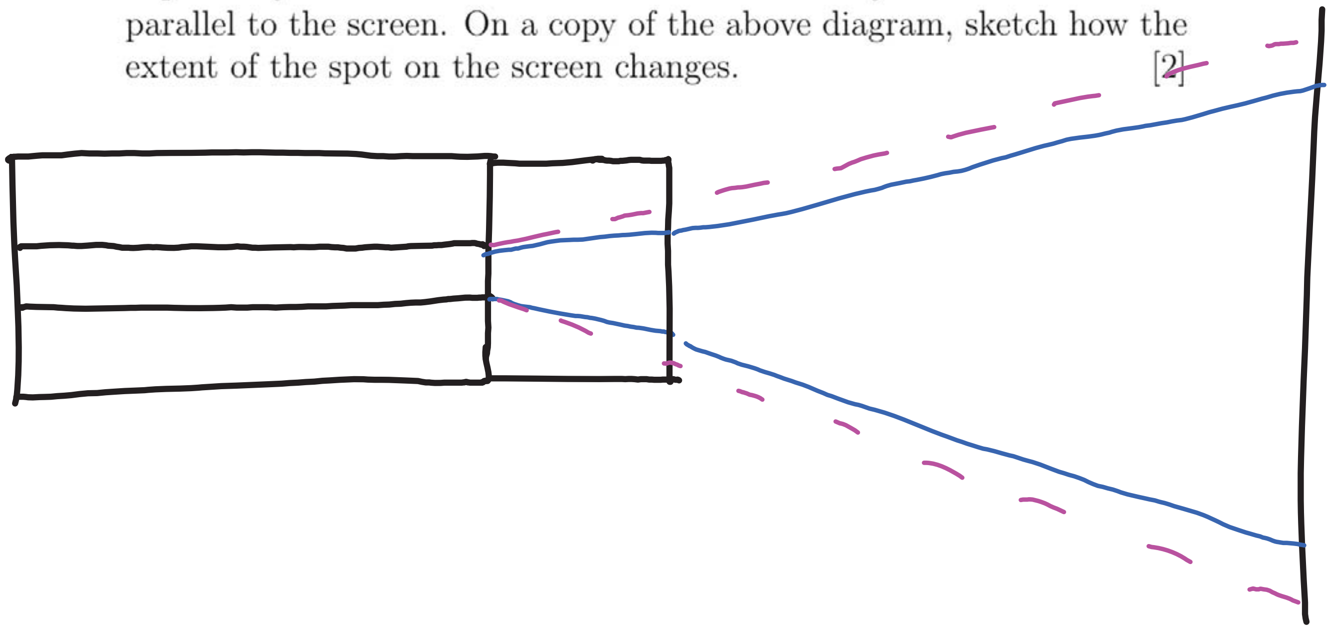
$= n_{core}\sqrt{1-\left(\frac{n_{core}}{n_{clad}}\right)^2}$

$$\theta_{max} = \arcsin\left[n_{core}\sqrt{1-\left(\frac{n_{core}}{n_{clad}}\right)^2}\right]$$

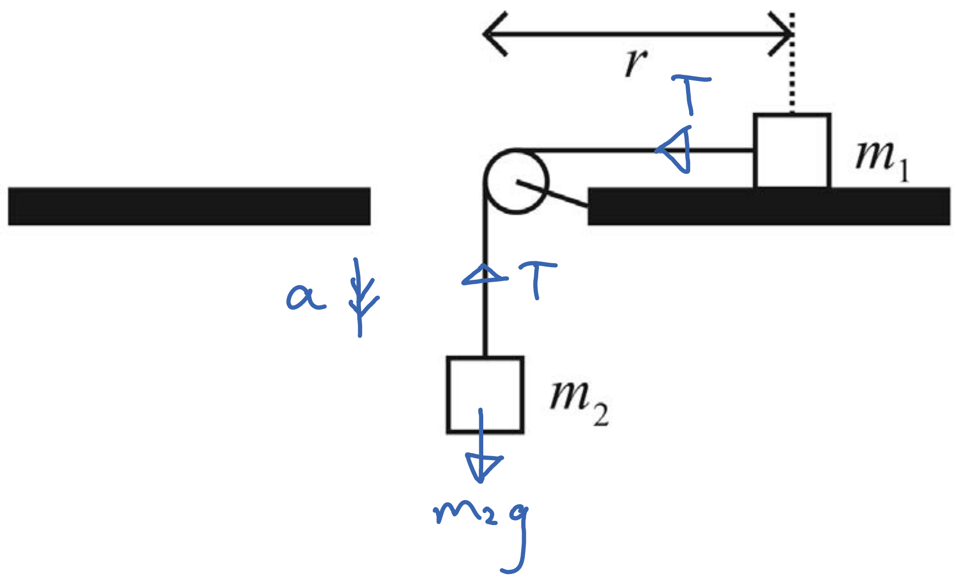


d) In an experiment, light is transmitted along the glass fibre before leaving at a perfectly vertical end. A screen is placed a long distance away from the end of the fibre and a uniform circular spot is seen as shown above.

A small glass tank containing water is placed in front of the beam so it perfectly touches the end of the fibre. You may assume the tank is parallel to the screen. On a copy of the above diagram, sketch how the extent of the spot on the screen changes. [2]



19 Two masses m_1 and m_2 are connected by a massless, non-extensible string supported by a massless pulley attached to a table with a hole in the middle; see sketch below.



a) Assuming no friction, derive an expression for the acceleration of the masses and for the tension of the string.

[2]

$1: [F = ma] \leftarrow$ $\qquad$ $2: [F = ma] \downarrow$

$T = m_1 a$ $\qquad$ $m_2 g - T = m_2 a$

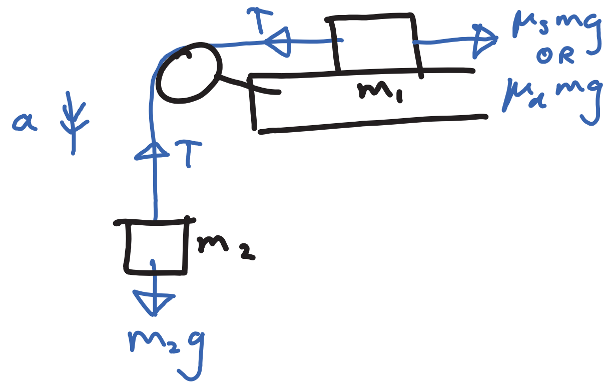
(1) in (2): $m_2 g - m_1 a = m_2 a$

$$a = \frac{m_2 g}{m_1 + m_2}$$

In (1): $T = \dfrac{m_1 m_2 g}{m_1 + m_2}$

Now and for the rest of this question, consider friction acting on the table but not on the pulley. Friction force F_{fr} is proportional to the mass's weight; $F_{fr} = \mu_s mg$ or $F_{fr} = \mu_d mg$ depending whether the mass is at rest (μ_s = static friction coefficient) or in motion (μ_d = dynamic friction coefficient). Both coefficients are known.

b) Derive expressions for the acceleration of the masses and for the tension of the string. What condition needs to be satisfied for m_1 to accelerate? [3]



1: $[F = ma] \leftarrow$

$$T - \mu_d m g = m_1 a$$

2: $[F = ma] \downarrow$

$$m_2 g - T = m_2 a$$

(1) + (2): $m_2 g - \mu_d m_1 g = (m_2 + m_1) a$

$$a = \frac{g(m_2 - m_1 \mu_d)}{m_2 + m_1}$$

In (2): $T = m_2 g - m_2 g \frac{(m_2 - m_1 \mu_d)}{m_2 + m_1}$

$$= m_2 g \left(1 + \frac{m_1 \mu_d - m_2}{m_1 + m_2}\right)$$

For acceleration, $T > \mu_s m_1 g$ and $T < m_2 g$

$\Rightarrow \mu_s m_1 g < T < m_2 g$

$\therefore \mu_s m_1 < m_2$