Wednesday, 4 November 2015

Time allowed: 2 hours

For candidates applying to Physics, Physics and Philosophy, Engineering, or Materials

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There are two Sections (A and B) to this test, each carrying equal weight.

Section A: Mathematics for Physics Q1-Q10 [50 Marks]

Section B: Physics Q11-Q21 [50 Marks]

Answers should be written on the question sheet in the spaces provided, and you should attempt as many questions as you can from each Section.

The numbers in the margin indicate the marks expected to be assigned to each question. You are advised to divide your time according to the marks available, and to spend equal effort on Sections A and B.

No calculators, tables, or formula sheets may be used.

Answers in Section A should be given exactly and in simplest terms unless indicated otherwise.

Numeric answers in Section B should be calculated to 2 significant figures unless indicated otherwise.

Do NOT turn over until told that you may do so.

Section A

1. Expand $(2x + x^2)^5$ in powers of x. [3]

$$(2x+x^2)^5 = (2x)^5 + 5(2x)^4(x^2)^1 + 10(2x)^3(x^2)^2 + 10(2x)^2(x^2)^3 + 5(2x)^1(x^2)^4 + (x^2)^5$$

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$= \underline{32x^5 + 80x^6 + 80x^7 + 40x^8 + 10x^9 + x^{10}}$$

2. Solve for x in the equation $\log_2 x + \log_4 16 = 2$. [3]

$$\log_2 x + 2 = 2$$

$$\log_2 x = 0$$

$$x = 2^0$$

$$x = 1$$

3. Evaluate the sum

$$\sum_{n=1}^{5} \left(\frac{1}{3}\right)^n.$$

What is the sum if the number of terms tends to infinity? [4]

Geometric: $a = \frac{1}{3}$, $r = \frac{1}{3}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^5\right]}{1-\frac{1}{3}} = \frac{\frac{1}{3} \times \frac{242}{243}}{\frac{2}{3}} = \underline{\frac{121}{243}}$$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \underline{\frac{1}{2}}$$

4. Evaluate the integral

$$\int_4^6 (2x-6)[(x-4)(x-2)]^{1/2}dx.$$

[5]

$$= \int_4^6 (2x-6)(x^2-6x+8)^{1/2}dx$$

$$= \left[\frac{2}{3}(x^2-6x+8)^{3/2}\right]_4^6$$

$$= \frac{2}{3}\left[(36-36+8)^{3/2} - (16-24+8)^{3/2}\right]$$

$$= \frac{2}{3}(8^{3/2}-0)$$

$$= \frac{2}{3}(2\sqrt{2})^3$$

$$= \frac{2}{3}\cdot 8 \cdot 2\sqrt{2}$$

$$= \frac{32\sqrt{2}}{3}$$

5. For what values of m does $4x^2 + 8x - 8 = m(4x - 3)$ have no real solutions? [4]

$4x^2 + 8x - 8 - 4mx + 3m = 0$

$4x^2 + (8 - 4m)x + (3m - 8) = 0$

$a\,x^2 \quad + \quad b\,x \quad + \quad c \quad = 0$

$\Delta = b^2 - 4ac$

$= (8 - 4m)^2 - 4(4)(3m - 8)$

$= 64 - 64m + 16m^2 - 48m + 128$

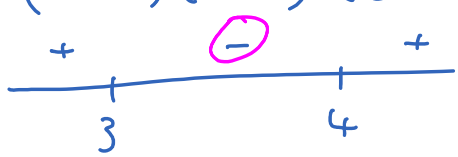
$= 16m^2 - 112m + 192$

$= m^2 - 7m + 12$

$= (m-4)(m-3)$

For no real solutions, $\Delta < 0$

$(m-4)(m-3) < 0$



$3 < m < 4$

6. An unbiased coin is tossed 3 times. Each toss results in a "head" or "tail". What is the probability

(a) of two or more tails in succession?

(b) that two consecutive toss results are the same?

(c) that if any one of the toss results is known to be a tail, that all of the tosses resulted in tails? [4]

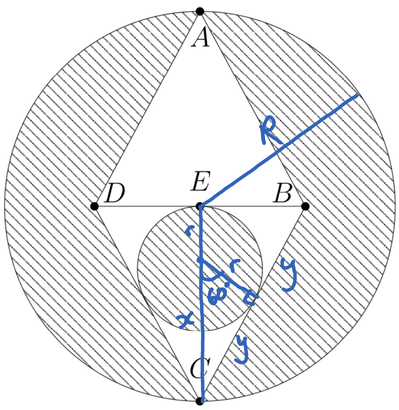
a) TTH , HTT, TTT : $3 \times \left(\frac{1}{2}\right)^3 = \frac{3}{8}$

b) $\left.\begin{matrix} TTH & THH \\ HHT & HTT \end{matrix}\right\}$ $4 \times \left(\frac{1}{2}\right)^3 = \frac{1}{2}$

c) $P(\text{all } T \mid \text{one } T) = \dfrac{P(\text{all } T \cap \text{one } T)}{P(\text{one } T)} = \dfrac{(1/2)^3}{1 - (1/2)^3}$ ← all H

$= \dfrac{1/8}{7/8} = \dfrac{1}{7}$

7. In the figure below, which is drawn only approximately to scale, the lengths of line segments AB, BC, CD, DA, and BD are equal. E is at the centre of the larger circle. The smaller circle is tangent to BC, CD, and BD, and has radius r. Derive an exact expression for the area of the shaded region in terms of r. [6]



$\cos 60 = \frac{r}{x}$

$x = 2r$

$\tan 60 = \frac{y}{r}$

$y = \sqrt{3}r$

$R = x + r$
$= 3r$

Area of $\triangle DCB = \frac{1}{2}(2y)R = \frac{1}{2} \cdot 2\sqrt{3}r \cdot 3r = 3\sqrt{3}r^2$

Area of big circle $= \pi R^2 = \pi(3r)^2 = 9\pi r^2$

Shaded area $=$ big circle $-$ 2(triangle) $+$ small circle

$= 9\pi r^2 - 2 \times 3\sqrt{3}r^2 + \pi r^2$

$= 10\pi r^2 - 6\sqrt{3}r^2$

$= 2r^2(5\pi - 3\sqrt{3})$

8. Find the slopes and y-intercepts of the straight lines that are tangent and normal to the circle $(x+3)^2 + (y-3)^2 = 17$ at the point $(1, 2)$. [5]

$$2(x+3) + 2(y-3)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x+3}{3-y}$$

At $(1,2)$, $\frac{dy}{dx} = \frac{1+3}{3-2} = 4$

Tangent:

$m = 4$

$y = mx + c$

$2 = 4(1) + c$

$c = -2$

Normal:

$m = -1/4$

$y = mx + c$

$2 = -\frac{1}{4}(1) + c$

$c = 9/4$

9. By sketching the function below, or otherwise, find what values of y the function takes when x can take any real value.

$$y = -\left(\frac{8}{x^2 - 4}\right) - 3$$

[8]

Limits: As $x \to \infty$, $y \to -3$ from below

$x \to -\infty$, $y \to -3$ from below

Asymptotes: $x^2 - 4 = 0$

$x = \pm 2$

Turning points: $y = -8(x^2 - 4)^{-1} - 3$

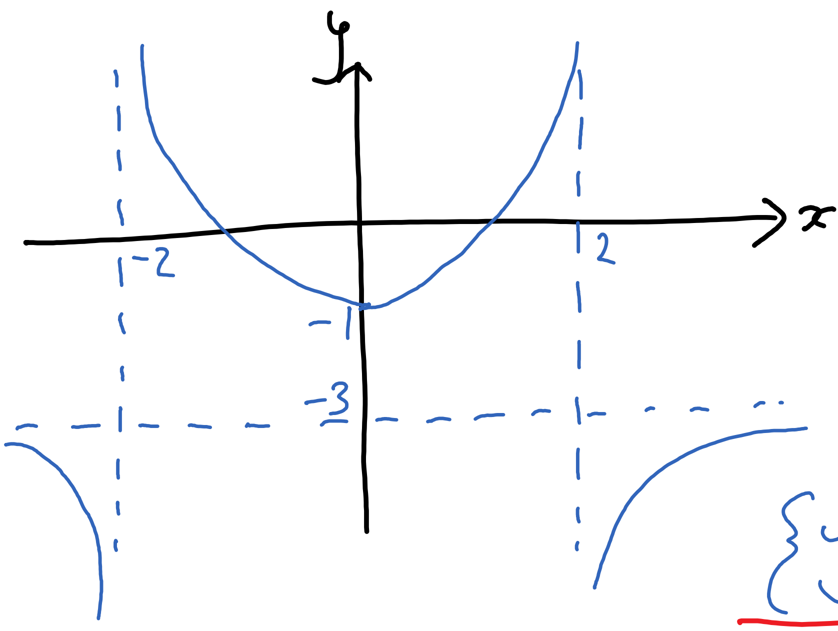
$$\frac{dy}{dx} = 8(x^2-4)^{-2} \cdot 2x = 0$$

$x = 0$ or $\frac{1}{(x^2-4)^2} = 0$

no solution

When $x = 0$, $y = -\left(\frac{8}{-4}\right) - 3$

$= -1$



$\{y \geqslant 1\} \cup \{y < -3\}$

10. For what values of x are the following inequalities satisfied?

$$-1 < \frac{3x+4}{x-6} < 1$$

[8]

$-1(x-6)^2 < (3x+4)(x-6)$ AND $(3x+4)(x-6) < (x-6)^2$

$-x^2+12x-36 < 3x^2-14x-24$

$0 < 4x^2-26x+12$

$0 < 2x^2-13x+6$

$0 < (2x-1)(x-6)$

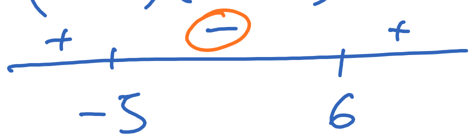
$x < \frac{1}{2}$ OR $x > 6$

$3x^2-14x-24 < x^2-12x+36$

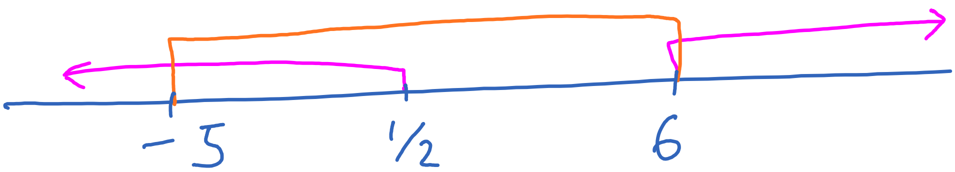
$2x^2-2x-60 < 0$

$x^2-x-30 < 0$

$(x-6)(x+5) < 0$



$-5 < x < 6$



$$-5 < x < \frac{1}{2}$$

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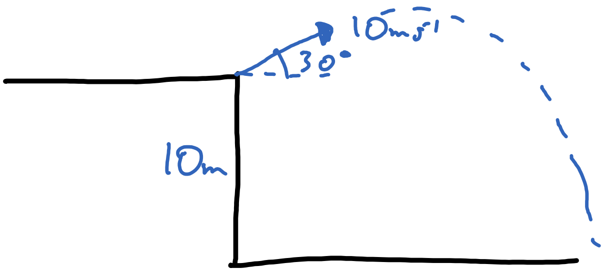
Section B

11. A ball is thrown at an angle of 30° up from the horizontal, at a speed of 10 m/s, off the top of a cliff which is 10 metres high above a flat beach. How long does it take for the ball to hit the beach below? You may assume that the acceleration due to gravity is 10 m/s^2, and that air resistance can be neglected. [4]



$+\downarrow$

$s = 10\text{m}$

$u = -10\sin 30 = -5\text{ms}^{-1}$

$v = \times$

$a = 10\text{ms}^{-2}$

$t = ?$

$$\left[s = ut + \frac{1}{2}at^2\right]$$

$$10 = -5t + \frac{1}{2}(10)t^2$$

$$0 = 5t^2 - 5t - 10$$

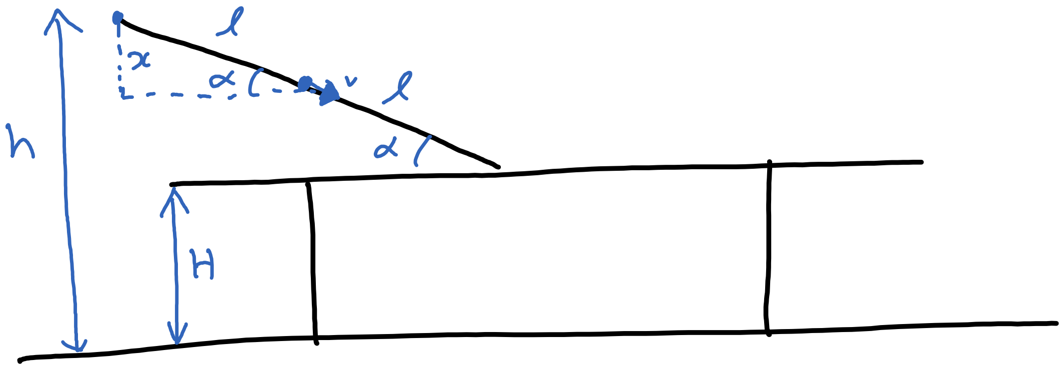
$$= t^2 - t - 2$$

$$= (t-2)(t+1)$$

$$t = -1 \text{ or } 2$$

But $t > 0$ $\therefore$ $t = 2\text{s}$

12. An ice cube slides down a frictionless slope, which is at an angle α to the horizontal. The slope sits on a horizontal table of height H above the ground. If the ice cube is released from rest at height h above the ground, what is the speed of the cube when it is half way down the slope? [3]



Using similar triangles,

$$\frac{x}{l} = \frac{h-H}{2l}$$

$$x = \frac{h-H}{2}$$

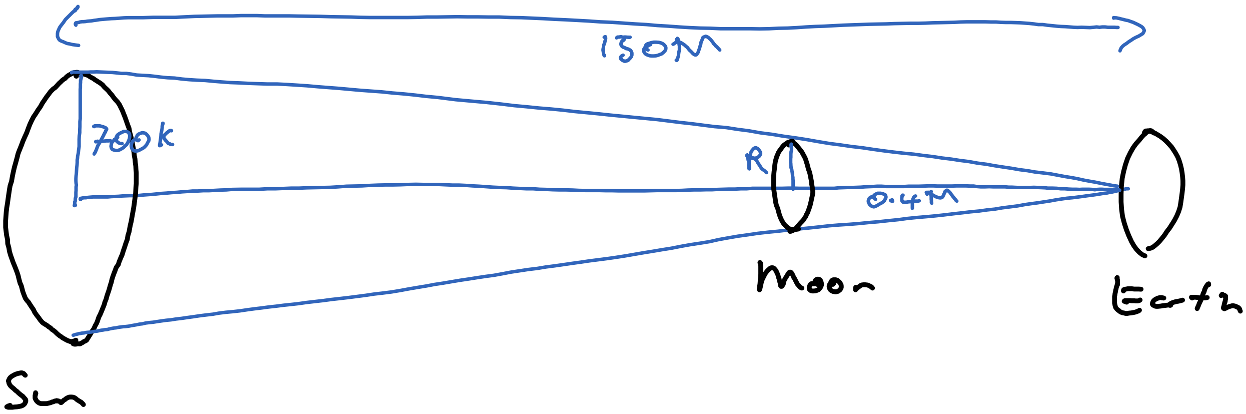
Conservation of energy:

$$\frac{1}{2}mv^2 = mgx$$

$$v = \sqrt{2gx}$$

$$= \sqrt{g(h-H)}$$

13. Using your knowledge of solar eclipses, estimate the radius of the Moon. You may assume the radius of the Sun is $700,000$ km, the distance from the Earth to the Sun is $150,000,000$ km, and the distance from the Earth to the centre of the Moon is $400,000$ km. [3]



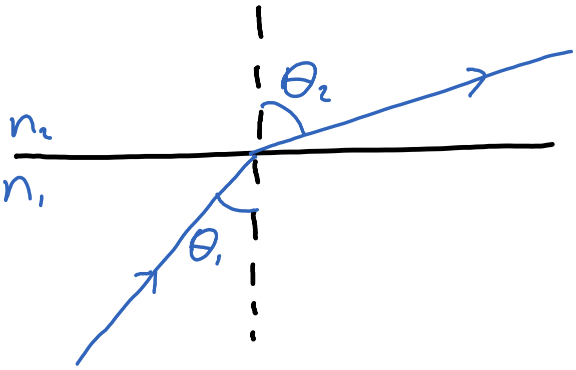
Using similar triangles,

$$\frac{R}{700} = \frac{0.4}{150}$$

$$R = \frac{700 \times 0.4}{150} = \frac{28}{15}$$

$\therefore$ Radius $\simeq$ 2000 km

14. Write an expression relating the angle of incidence θ_1 and angle of refraction θ_2 of a ray of light travelling from one optically transparent material with index of refraction n_1 to another with index of refraction n_2. Sketch a diagram of the ray of light, clearly labelling the angles and indices of refraction. Under what conditions and for what angles of incidence is light reflected completely at the boundary? [3]



$n_1 \sin\theta_1 = n_2 \sin\theta_2$

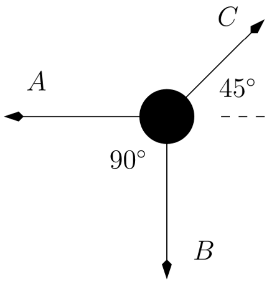
Total internal reflection occurs when $n_2 < n_1$

for $\theta_1 > \theta_c$, where

$n_1 \sin\theta_c = n_2$

$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$

15. Consider a mass and three strings, all lying on a horizontal table. The strings exert forces outwards on the mass as shown below. The mass does not move. What is the force on string C in terms of the force on string A? What is the relationship between the force exerted by string A and the force exerted by string B? [3]



$\rightarrow \quad A = C\cos 45$

$C = \frac{2A}{\sqrt{2}} = \sqrt{2}A$

$\uparrow \quad B = C\cos 45$

$B = A$

16. A non-rotating ball of mass 2 kg slides on a smooth, frictionless, horizontal surface at a speed of 1 m/s. It collides elastically and head-on with a stationary ball of mass 1 kg. What are the speeds of the two balls after the collision? [4]

Cons. mom: $2(1) + 0 = 2v_1 + v_2$

$2 = 2v_1 + v_2 \quad (1)$

Coef. of restitution: $e = 1$ for elastic collision

separation speed = e (approach speed)

$v_2 - v_1 = 1(1-0)$

$v_2 - v_1 = 1 \quad (2)$

$(1)-(2)$: $1 = 3v_1$

$v_1 = 0.3ms^{-1}$

(2): $v_2 = 1 + v_1$

$= 1.3 ms^{-1}$

17. A small boat floats on the sea. It encounters waves of the form

$$y(x,t) = A\sin(kx - \omega t)$$

where $y(x,t)$ is the height of the wave at position x and time t, and k and ω are constants. The waves have a wavelength of 10 metres, amplitude of 0.5 metres, and travel at a horizontal speed of 2 metres per second. What is the maximum vertical velocity of the boat? [4]

Given $\lambda = 10$, $A = 0.5$, $v = 2$

$v = f\lambda$

$f = \frac{2}{10} = 0.2\,\text{Hz}$

$= 0.2\,\text{Hz}$

$\omega = 2\pi f$

$= 2\pi \times 0.2$

$= 0.4\pi$

max vel $= \omega A$

$= 0.4\pi \times 0.5$

$= 0.6\,\text{ms}^{-1}$

18. A garden hose with a cross sectional area A ejects water at a rate of x, usually measured in units of $\mathrm{m^3\,s^{-1}}$.

(a) What is the speed of the water leaving the nozzle?

(b) The water hits a wall close to the end of the garden hose, perpendicular to the direction of flow. What is the force on the wall if (i) the water falls to the ground when it hits the wall, or (ii) the water rebounds horizontally? You may assume the density of water is ρ, usually measured in units of $\mathrm{kg\,m^{-3}}$. [5]

a) speed: $\mathrm{ms^{-1}} = \dfrac{\mathrm{m^3 s^{-1}}}{\mathrm{m^2}} \Rightarrow \text{speed} = \dfrac{x}{A}$

b) $F = \dfrac{\Delta p}{\Delta t} = \dfrac{m\Delta v}{\Delta t}$; $\dfrac{m}{\Delta t} = \rho x$ $(m = \rho V)$

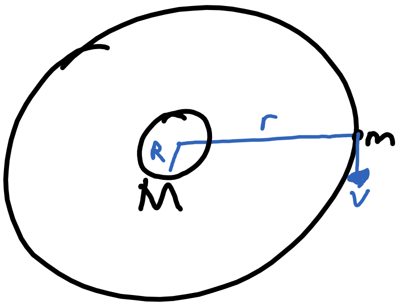
i) $\Delta v = \dfrac{x}{A} \Rightarrow F = \rho x \cdot \dfrac{x}{A} = \dfrac{\rho x^2}{A}$

ii) $\Delta v = \dfrac{x}{A} - \left(-\dfrac{x}{A}\right) = \dfrac{2x}{A} \Rightarrow F = \dfrac{2\rho x^2}{A}$

19. Consider a circular orbit of radius r around the Earth. If the Earth's mass is M and radius is R, and Newton's gravitational constant is G, derive an expression for the speed that is required for a stable orbit.

Assuming that the radius of the Earth is 6400 km, and the gravitational acceleration g at the surface of the Earth is 10 m/s^2, calculate the speed required for a stable orbit around the equator at sea level. [5]

What physical reasons make it difficult for a satellite to maintain a circular orbit around the equator at sea level, even if one can ignore the atmosphere? [1]



$$\frac{GM\cancel{m}}{r^{\cancel{2}}} = \frac{\cancel{m}v^2}{\cancel{r}}$$

$$v = \sqrt{\frac{GM}{r}}$$

For $\frac{GM}{r^2} = g = 10$, $r = R = 6.4\times10^6$,

$$v = \sqrt{\frac{GM}{R^2}\cdot R} = \sqrt{10 \times 6.4\times10^6} = 8000\ \mathrm{ms^{-1}}$$

• It would come across mountains

20. The energy levels of hydrogen are given by the expression

$$E_n = -\left(\frac{R}{n^2}\right)$$

where R is the Rydberg constant, and n is a positive integer.

Neutral hydrogen atoms are excited to a state with $n = 10$. The atoms then de-excite, emitting light before settling to the ground state ($n = 1$).

(a) What is the shortest wavelength of light emitted?

(b) What is the longest wavelength emitted?

(c) How many emission lines may be observed?

[Leave answers as fractions multiplied by powers of R, Planck's constant h, and the speed of light c as needed.] [7]

$$\Delta E = \frac{hc}{\lambda}$$

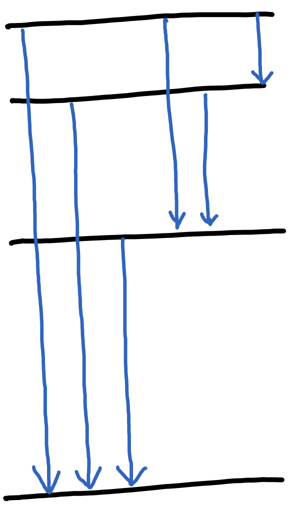
a) $$\frac{hc}{\lambda} = \frac{R}{1^2} - \frac{R}{10^2} = \frac{99R}{100}$$

$$\lambda = \frac{100hc}{99R}$$

b) $$\frac{hc}{\lambda} = \frac{R}{9^2} - \frac{R}{10^2} = \frac{19R}{8100}$$

$$\lambda = \frac{8100hc}{19R}$$

c)



$$9 + 8 + \dots + 1 = 45$$

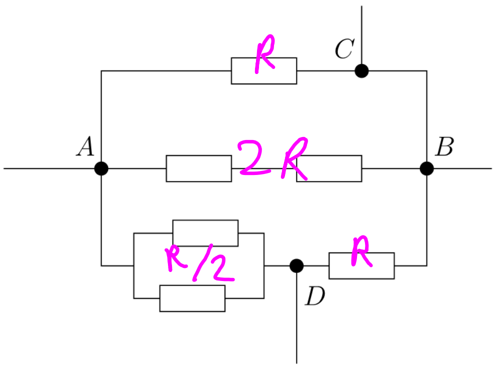
21. Consider the resistor array below. All the resistors are identical, with resistance R.

(a) Calculate the total resistance between A and B.

(b) If a potential difference V is applied between A and B, calculate the power dissipated in the resistor between B and D.

(c) Calculate the total resistance between C and D.

[Leave answers as fractions multiplied by powers of R and V as needed.] [8]



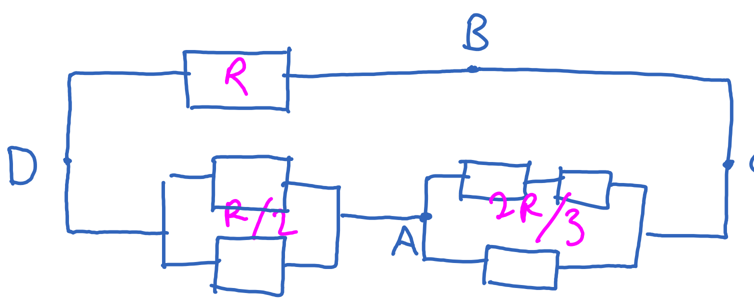
a) $\frac{1}{R_T} = \frac{1}{R} + \frac{1}{2R} + \frac{1}{3R/2}$

$= \frac{6+3+4}{6R}$

$R_T = \frac{6R}{13}$

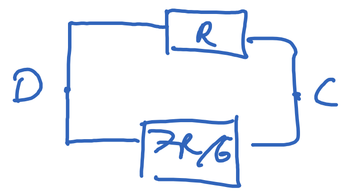
b) $V_{BD} = V \cdot \frac{R}{3R/2} = \frac{2V}{3}$ $\quad P = \frac{V^2}{R} = \frac{\left(\frac{2V}{3}\right)^2}{R} = \frac{4V^2}{9R}$

c)



$\frac{R}{2} + \frac{2R}{3} = \frac{7R}{6}$

$\equiv$



$\frac{1}{R_T} = \frac{1}{R} + \frac{6}{7R} = \frac{13}{7R}$

$R_T = \frac{7R}{13}$