महाराष्ट्र वनसेवा मुख्य परीक्षा - २०१२

2012

MATHEMATICS (Optional)

गणित (वैकल्पिक)

100276

Time : 3 hours **Maximum Marks : 200**

Note :

(i) *Answers must be written in English.*

(ii) *Question* **No. 1** *is* **compulsory**. *Of the remaining questions, attempt* **any Four** *selecting one question from* **each section**.

(iii) *Figure to the* **RIGHT** *indicates marks of the respective question.*

(iv) *Number of optional questions upto the prescribed number in the order in which they have been solved will only be assessed. Excess answers will not be assessed.*

(v) *Candidates should not write roll number, any name (including their own), signature, address or any indication of their identity anywhere inside the answer book otherwise they will be penalised.*

1. Answer *any four* of the following :

(a) A company has three operational departments (weaving, processing and packing) with a capacity to produce three different types of cloth namely suiting, shirting and woolens yielding the profit of Rs. 2, Rs. 4 and Rs. 3 per metre respectively. One metre suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. One metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing while one metre of woolen requires 3 minutes in each department. In a week total run time is 60, 40, 80 hours of weaving, processing and packing departments respectively. Formulate the linear programming model to find the product mix to maximize the profit. Solve it using simplex method. **10**

(b) Examine the convergence and absolute convergence of the series **10**

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{(n^2+1)}$$

(c) A particle moves with a central acceleration μr^{-7} and starts from an apse at a distance 'a' with a velocity equal to the velocity which would be acquired by the particle travelling from rest at an infinity to the apse. Show that the equation of its orbit is $r^2 = a^2 \cos 2\theta$. **10**

(d) Write a computer program in C for evaluation of the integral **10**

$\int_{2.5}^{7.8} \frac{x^3 + 2x^2 + 5x + 6}{x^2 - 3x + n} dx$, using Simpson's rule. Select n = 100.

P.T.O.

(e) If (i) $u_k(x) \in c$, $a \le x \le b$, $k=1, 2, 3, ...$ **10**

(ii) $f(x) = \sum_{k=1}^{\infty} u_k(x)$, uniformly in $a \le x \le b$, then prove that

$$\int_a^b f(x)\,dx = \sum_{k=1}^{\infty} \int_a^b u_k(x)\,dx$$

SECTION - A

2. (a) Prove the following theorems :

(i) Let $G=\langle a \rangle$ be a cyclic group order n and H be a subgroup of G generated by a^m, $m \le n$. Then, **10**

$$O(H) = \frac{n}{\gcd(m, n)}$$

(ii) Any finite group is isomorphic to a permutation group. **10**

(b) Verify Cayley - Hamilton theorem for the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and hence find its inverse. **20**

3. (a) Let $f: G \to H$ be a group homomorphism of a group G onto another group H and let ker (f) be the kernel of f. Then prove that $\frac{G}{\ker(f)} \simeq H$ **20**

(b) Evaluate Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$, and determine whether the Eigen vectors are orthogonal. **20**

SECTION - B

4. (a) If $f(x)$ is continuous in [a, b], differentiable in (a, b) and $f(a)=f(b)$, then prove that there exists at least one $c \in (a, b)$ such that $f^1(c)=0$. **20**
Plot the graph of the function $f(x) = 1-x^{2/3}$. Does this function satisfy all conditions above theorem for $x \in [-1, 1]$. Verify the above theorem for the function $f(x)=x(x-2)\,e^{3x/4}$, $0 \le x \le 2$.

(b) Use double integration to find the area bounded by three lines $x+2y=2$, $y-x=1$ and $2x+y=7$

Also, evaluate $\int_{-2}^{1} \int_{x^2+4x}^{3x+2} dy\, dx$ by changing order of integration. 10

5. (a) (i) The temperature T at any point (x, y, z) in a space is given by $T(x, y, z)=8xyz^2$. Find the highest temperature on the surface of the sphere $x^2+y^2+z^2=36$. 10

(ii) Are the functions u, v and w defined as $u(x, y, z)=x+y+z$; $v(x, y, z)=x^3+y^3+z^3-3xyz$ and $w(x, y, z)=x^2+y^2+z^2-xy-yz-zx$ functionally related ? If yes, find the relationship between them. 10

(b) Use triple integration to obtain the volume of the solid surrounded by the surface 20

$$\left(\frac{x}{7}\right)^{2/3}+\left(\frac{y}{5}\right)^{2/3}+\left(\frac{z}{8}\right)^{2/3}=1$$

SECTION - C

6. (a) Does the equation $16x^2-24xy+9y^2-6x-8y-1=0$ represent a parabola ? If yes, reduce it to a simplest form and obtain : 20

(i) equation of its axis,

(ii) co-ordinates of vertex,

(iii) equation of tangent at vertex

(iv) co-ordinates of focus and

(v) the length of its latus rectum

(b) (i) Prove that the vector $\overline{A}=(6xy+z^3)\,i+(3x^2-z)\,j+(3xz^2-y)\,k$ is irrotational. Find scalar function $f(x, y, z,)$ such that $\overline{A}=\overline{\nabla} f$. 10

(ii) Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx-xy\,dy$ where C is the boundary of the annulus region enclosed by $x^2+y^2=1$ and $x^2+y^2=9$. 10

7. (a) (i) A sphere of radius 5 passes through the origin and cuts the coordinate axes in A, B, C. Prove that the locus of the foot of perpendicular from origin to the plane ABC is given by $(x^2+y^2+z^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=100$ 10

(ii) Obtain equation of right circular cone with vertex at (1, 2, −3), semivertical angle $\cos^{-1}\frac{1}{\sqrt{3}}$ and the line $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z+3}{-2}$ as its axis. 10

P.T.O.

(b) (i) Verify Stokes' theorem for $\overline{A} = y^2 i + xyj - zxk$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2,\ z \geq 0$. **10**

(ii) For any function $f(x, y, z)$, prove that $\nabla \times \nabla f = 0$. **10**

SECTION - D

8. (a) Solve the following differential equations :

(i) $x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = x + \log x$ **10**

(ii) $2 \frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 5 \sin (2x + y)$ **10**

(b) (i) Evaluate the integral $\int_1^2 \left(\frac{x+2}{x-1}\right) dx$ by Gauss quadrature method for n=3. **10**

(ii) Use Runge-Kutta method to obtain y (2) if $\frac{dy}{dx} = \frac{x+1}{y}$ and y (1)=1. Select h=0.25. **10**

9. (a) Solve the following differential equations

(i) $(2x+5)^2 \frac{d^2 y}{dx^2} - 6\ (2x+5) \frac{dy}{dx} + 8y = 6x$ **10**

(ii) $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$ **10**

(b) (i) Evaluate the integral $\int_1^3 3x^2 dx$ by using Simpson rule. Take n=10. Calculate the percentage error of the numerical result relative to the analytic one. **10**

(ii) Use Euler method to obtain y (2) if $\frac{dy}{dx} = \frac{x+1}{y}$, $y(1) = 1$. **10**

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