A

2013 – MS

Test Paper Code : MS

Time : 3 Hours **Maximum Marks : 100**

INSTRUCTIONS

1. This question-cum-answer booklet has **32** pages and has **30** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions,** provided on Page No. **4.** Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **2 (Two)** marks.
   (b) For each wrong answer, you will be awarded **-0.5 (Negative 0.5)** mark.
   (c) Multiple answers to a question will be treated as a wrong answer
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the fill in the blank type and descriptive type questions only in the space provided after each question. No negative marks for fill in the blank type questions.
6. Do not write more than one answer for the same question. In case you attempt a fill in the blank or a descriptive question more than once, please cancel the answer(s) you consider wrong. Otherwise the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, cellular phone and electronic gadgets in any form are NOT allowed. Non Programmable calculator is allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to Special instructions/useful data on the reverse.

2013 – M

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER							

Name :

Test Centre :

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

Signature of the Candidate

I have verified the information filled by the Candidate above.

Signature of the Invigilator

Special Instructions / Useful Data

$\mathbb{R}$:	Set of all real numbers
$\mathbb{Q}$:	Set of all rational numbers
$P(A)$:	Probability of an event A
i.i.d.:	independent and identically distributed
$\text{Exp}(\lambda)$:	The exponential distribution with density $f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x}, & \text{if } x>0,\\ 0, & \text{otherwise,}\end{cases}\quad \lambda>0$
$N(\mu,\sigma^2)$:	Normal distribution with mean $\mu\in\mathbb{R}$ and variance $\sigma^2>0$
t_n:	Central Student's t-distribution with n degrees of freedom
χ^2_n:	Central Chi-square distribution with n degrees of freedom
$\chi^2_{n,\alpha}$:	a constant such that $P(W>\chi^2_{n,\alpha})=\alpha$, where W has χ^2_n
$\Phi(x)$:	Cumulative distribution function of $N(0,1)$
$\varphi(x)$:	Density function of $N(0,1)$
$A^c=$	Complement of the event A
$\Phi(2.33)=$	0.99
$E(X)$:	Expectation of a random variable X
$\text{Var}(X)$:	Variance of a random variable X
$\text{Corr}(X,Y)$:	Correlation coefficient between random variables X and Y
$\Gamma(\alpha)=$	$\int_0^\infty e^{-x}x^{\alpha-1}\,dx,\quad \alpha>0$

IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-10 (objective questions) carry *two* marks each, questions 11-20 (fill in the blank questions) carry *three* marks each and questions 21-30 (descriptive questions) carry *five* marks each.**
- **The marking scheme for the objective type question, is as follows:**
  (a) For each correct answer, you will be awarded **2 (Two)** marks.
  (b) For each wrong answer, you will be awarded **-0.5 (Negative 0.5)** mark.
  (c) Multiple answers to a question will be treated as a wrong answer.
  (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
  (e) Negative marks for objective part will be carried over to total marks.
- **There is no negative marking for fill in the blank questions.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 4 only.**

Objective Questions

Q.1 Let E and F be two events with $P(E)=0.7, P(F)=0.4$ and $P(E \cap F^c)=0.4$. Then $P(F \mid E \cup F^c)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

Q.2 Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$. Then $\lim_{n\to\infty} \frac{e^{a_n^2}+a_n}{4}$ is equal to

(A) ∞ (B) $\frac{e^{1/4}}{4}+\frac{1}{8}$ (C) $\frac{e^{1/4}}{4}$ (D) $\frac{1}{4}$

Q.3 Let $f:[0,\infty)\to[0,\infty)$ be a twice differentiable and increasing function with $f(0)=0$. Suppose that, for any $t\geq 0$, the length of the arc of the curve $y=f(x)$, $x\geq 0$ between $x=0$ and $x=t$ is $\frac{2}{3}\left[(1+t)^{3/2}-1\right]$. Then $f(4)$ is equal to

(A) $\frac{11}{3}$ (B) $\frac{13}{3}$ (C) $\frac{14}{3}$ (D) $\frac{16}{3}$

Q.4 Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y)=\begin{cases} \dfrac{\sin(2(x^2+y^2))}{x^2+y^2}\, e^{3x\sin\left(\frac{4}{y}\right)}, & \text{if } (x,y)\neq(0,0), \\ \alpha, & \text{if } (x,y)=(0,0), \end{cases}$$

where α is a real constant. If f is continuous at $(0,0)$, then α is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Q.5 Let A be a 3×3 real matrix with eigenvalues 1, 2, 3 and let $B = A^{-1} + A^2$. Then the trace of the matrix B is equal to

(A) $\frac{91}{6}$ (B) $\frac{95}{6}$ (C) $\frac{97}{6}$ (D) $\frac{101}{6}$

Q.6 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with variance 1. Then $\lim_{n\to\infty} P\left(\frac{(X_1-X_2)+(X_3-X_4)+\cdots+(X_{2n-1}-X_{2n})}{\sqrt{n}} \le x\right)$ is equal to

(A) $\Phi(x)$ (B) $\Phi(2x)$ (C) $\Phi\left(x\sqrt{2}\right)$ (D) $\Phi\left(\frac{x}{\sqrt{2}}\right)$

Q.7 Let $X_1, X_2, \ldots, X_{100}$ be a random sample from a $N(2,4)$ population. Let $\overline{X} = \frac{1}{99}\sum_{i=1}^{99} X_i$, $S = \sqrt{\frac{1}{98}\sum_{i=1}^{99}(X_i - \overline{X})^2}$ and $W = \frac{X_{100}-2}{S}$. Then the distribution of W is

(A) χ^2_{98} (B) χ^2_{99} (C) t_{98} (D) t_{99}

Q.8 Let $X_1, X_2, \ldots, X_n, X_{n+1}$ be a random sample from a $N(\mu,1)$ population. If $\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $T = \frac{1}{2}(\overline{X_n} + X_{n+1})$, then for estimating μ

(A) T is unbiased and consistent

(B) T is biased and consistent

(C) T is unbiased and inconsistent

(D) T is biased and inconsistent

Q.9 Let X be an observation from a population with density

$$f(x)=\begin{cases}\lambda^2 x\, e^{-\lambda x}, & \text{if } x>0, \lambda>0,\\ 0, & \text{elsewhere.}\end{cases}$$

For testing $H_0:\lambda=2$ against $H_1:\lambda=1$, the most powerful test of size α is given by "Reject H_0 if $X > c$", where c is given by

(A) $\frac{1}{4}\chi^2_{4,\alpha}$ (B) $\frac{1}{4}\chi^2_{3,\alpha}$ (C) $\frac{1}{4}\chi^2_{2,\alpha}$ (D) $\frac{1}{4}\chi^2_{1,\alpha}$

Q.10 A continuous random variable X has the density

$$f(x)=2\,\varphi(x)\,\Phi(x),\ x\in\mathbb{R}.$$

Then

(A) $E(X)>0$ (B) $E(X)<0$

(C) $P(X\le 0)>0.5$ (D) $P(X\ge 0)<0.25$

Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question Number. Do not write anything else on this page.

Question Number	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		

FOR EVALUATION ONLY

Number of Correct Answers		Marks	(+)
Number of Incorrect Answers		Marks	(–)
Total Marks in Questions 1-10			( )

Fill in the blank questions

Q.11 If X has the probability density function

$$f(x)=\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x};\ x>0,\alpha>2,\ \text{then } \operatorname{Var}\left(\frac{1}{X}\right) \text{ is equal to}$$

Ans:

Q.12 Let the joint density function of (X,Y) be

$$f(x,y)=\begin{cases} c\,(x+y), & \text{if } -x<y<x,\ 0<x<1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the value of c is equal to

Ans:

Q.13 Let X be an observation from a population with density function $f(x)$. Then the power of the most powerful test of size $\alpha=0.19$ for testing

$$H_0: f(x)=\begin{cases} \frac{x}{2}, & \text{if } 0<x<2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{against } H_1: f(x)=\begin{cases} \frac{3x^2}{8}, & \text{if } 0<x<2, \\ 0, & \text{otherwise,} \end{cases}$$

is equal to

Ans:

Q.14 Bulbs produced by a factory F_i have lifetimes (in months) distributed as $\operatorname{Exp}\left(\frac{1}{3^i}\right)$ for $i=1,2,3$. A firm randomly procures 40% of its required bulbs from F_1, 30% from F_2 and 30% from F_3. A randomly selected bulb from the firm is found to be working after 27 months. The probability that it was produced by the factory F_3 is

Ans:

Q.15 Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x,\mu) = \begin{cases} e^{\mu - x}, & \text{if } x > \mu, \\ 0, & \text{otherwise,} \end{cases}$$

and let $X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$. Then $\left(X_{(1)} - \frac{2}{n}\log_e 5,\ X_{(1)}\right)$ is a ....% confidence interval for μ.

Ans:

Q.16 Ten percent of bolts produced in a factory are defective. They are randomly packed in boxes such that each box contains 3 bolts. Four of these boxes are bought by a customer. The probability, that the boxes that this customer bought have no defective bolt in them, is equal to

Ans:

Q.17 Let $f:\mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ \alpha x + \beta, & \text{if } x \in \mathbb{R}\text{-}\mathbb{Q}, \end{cases}$$

where α and β are real constants. If f is differentiable at $x = 1$ then the value of $3\alpha + \beta$ is equal to

Ans:

Q.18 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $|a_n| \leq \sqrt{n}$, $n = 1, 2, \ldots$. Then

$$\lim_{n\to\infty} \left\{ e^{\frac{a_n}{n}} + \sqrt{n}\sin\left(\sqrt{\frac{2}{n}}\right) \right\}$$

is equal to

Ans:

Q.19 Consider the linear system

$$x + y + 2z = \alpha$$
$$x + 4y + z = 4$$
$$3y - z = \gamma$$

in the unknowns x, y and z. If the above system always has a solution then the value of $\alpha + \gamma$ is equal to

Ans:

Q.20 The general solution of the differential equation $(x^4 - y)\,dx + (y^4 - x)\,dy = 0$ is equal to

Ans:

Descriptive questions

Q.21 Consider the matrix

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ \alpha & 0 & \beta \end{bmatrix}.$$

If P has eigenvalues 0 and 3 then determine the values of the pair (α, β).

Space for the answer

Space for the answer

Q.22 Let a function $f:[0,1]\to\mathbb{R}$ be continuous on $[0,1]$ and differentiable in (0, 1). If $f(0)=1$ and $[f(1)]^3+2f(1)=5$, then prove that there exists a $c\in(0,1)$ such that $f'(c)=\dfrac{2}{2+3\,[f(c)]^2}$.

Space for the answer

Space for the answer

Q.23 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that the series $\sum_{n=1}^{\infty} \log_e(1+a_n^4)$ converges.

Space for the answer

Space for the answer

Q.24 Let $D=\{(x,y)\in\mathbb{R}^2 : 0\le y\le x\le 1\}$ and let $f:D\to\mathbb{R}$ be defined by $f(x,y)=x^2-2xy+2,\ (x,y)\in D$. Then determine the maximum value of f in the region D.

Space for the answer

Space for the answer

Q.25 Let X, Y and Z be independent random variables with respective moment generating functions $M_X(t) = \frac{1}{1-t}$, $t < 1$; $M_Y(t) = e^{t^2/2} = M_Z(t)$, $t \in \mathbb{R}$. Let $W = 2X + Y^2 + Z^2$. Then determine the value of $P(W > 2)$.

Space for the answer

Space for the answer

Q.26 Ram rolls a pair of fair dice. If the sum of the numbers shown on the upper faces is 5, 6, 10, 11 or 12 then Ram wins a gold coin. Otherwise, he rolls the pair of dice once again and wins a silver coin if the sum of the numbers shown on the upper faces in the second throw is the same as the sum of the numbers in the first throw. What is the probability that he wins a gold or a silver coin?

Space for the answer

Space for the answer

Q.27 Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $[\theta, 2\theta]$, $\theta > 0$. Find the method of moments estimator and the maximum likelihood estimator of θ. Further find the bias of the maximum likelihood estimator.

Space for the answer

Space for the answer

Q.28 Let (X_1, Y_1), (X_2, Y_2),........ be a sequence of i.i.d. bivariate normal random variables with $E(X_1) = 75$, $E(Y_1) = 25$, $\text{Var}(X_1) = 36$, $\text{Var}(Y_1) = 16$ and $\text{Corr}(X_1, Y_1) = 0.25$. Let $\bar{U} = \frac{1}{n}\sum_{i=1}^{n}(X_i + Y_i)$. Find the minimum value of n so that $P(\bar{U} \leq 104) \geq 0.99$.

Space for the answer

Space for the answer

Q.29 The joint probability density function of (X,Y) is

$$f(x,y)=\begin{cases}\frac{1}{2}e^{(1-x-y)}, & \text{if } x+y>1,\ x>0,\ y>0,\\ 0, & \text{otherwise.}\end{cases}$$

Find the probability density function of X and $E(Y\mid X=x),\ x>0$.

Space for the answer

Space for the answer

Q.30 Suppose that F is a cumulative distribution function, where

$$F(x)=\begin{cases} 0, & \text{if} \quad x<0, \\ 1-e^{-x}, & \text{if} \quad 0\le x<1, \\ c, & \text{if} \quad 1\le x<2, \\ 1-e^{-x}, & \text{if} \quad x\ge 2. \end{cases}$$

i. Find all possible values of c.

ii. Find $P(0.5 \le X \le 2.5)$ and $P(X=1)+P(X=2)$.

Space for the answer

Space for the answer

Space for rough work

Space for rough work

Space for rough work

Space for rough work

Space for rough work

2013 - MS Objective Part (Question Number 1 – 10)	
Total Marks	Signature

Fill in the blanks Part and Descriptive Part					
Question Number	Marks		Question Number	Marks	
11			21		
12			22		
13			23		
14			24		
15			25		
16			26		
17			27		
18			28		
19			29		
20			30		
Total Marks in Fill in the blanks Part and Descriptive Part					

Total (Objective Part) :	
Total (Fill in the blanks Part and Descriptive Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	