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2010 – MA

Test Paper Code : MA

Time : 3 Hours **Max. Marks : 300**

INSTRUCTIONS

1. The question-cum-answer booklet has **40** pages and has **29** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the Name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer : (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **–2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone or electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

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READ INSTRUCTIONS ON THE SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name :

Test Centre :

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

Signature of the Candidate

I have verified the information filled by the Candidate above.

Signature of the Invigilator

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Special Instructions / Useful Data

R	:	The set of all real numbers
Q	:	The set of all rational numbers
N	:	The set $\{1,2,3,\ldots\}$ of all natural numbers
ϕ	:	The empty set
$E \setminus F$	:	The set $\{x \in E : x \notin F\}$, where E and F are sets
$\ln(x)$	:	The logarithm of x with respect to the base e

DO NOT WRITE ON THIS PAGE

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Which of the following conditions does NOT ensure the convergence of a real sequence $\{a_n\}$?

(A) $|a_n - a_{n+1}| \to 0$ as $n \to \infty$

(B) $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ is convergent

(C) $\sum_{n=1}^{\infty} n\,a_n$ is convergent

(D) The sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$ and $\{a_{3n}\}$ are convergent

Q.2 The value of $\iint_G \frac{\ln(x^2+y^2)}{x^2+y^2}\,dx\,dy$, where $G = \{(x,y) \in \mathbf{R}^2 : 1 \le x^2+y^2 \le e^2\}$, is

(A) π (B) 2π (C) 3π (D) 4π

Q.3 The number of elements of S_5 (the symmetric group on 5 letters) which are their own inverses equals

(A) 10 (B) 11 (C) 25 (D) 26

Q.4 Let S be an infinite subset of $\mathbf{R}$ such that $S \cap \mathbf{Q} = \phi$. Which of the following statements is true?

(A) S must have a limit point which belongs to $\mathbf{Q}$

(B) S must have a limit point which belongs to $\mathbf{R}\setminus\mathbf{Q}$

(C) S cannot be a closed set in $\mathbf{R}$

(D) $\mathbf{R}\setminus S$ must have a limit point which belongs to S

Space for rough work

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Q.5 Let $f:(1,4)\rightarrow \mathbf{R}$ be a uniformly continuous function and let $\{a_n\}$ be a Cauchy seq in $(1, 2)$. Let $x_n = a_n^2 f(a_n^2)$ and $y_n = \dfrac{1}{1+a_n^2} f(a_n^2)$, for all $n \in \mathbf{N}$. Which of the following statements is true?

(A) Both $\{x_n\}$ and $\{y_n\}$ must be Cauchy sequences in $\mathbf{R}$

(B) $\{x_n\}$ must be a Cauchy sequence in $\mathbf{R}$ but $\{y_n\}$ need not be a Cauchy sequence in $\mathbf{R}$

(C) $\{y_n\}$ must be a Cauchy sequence in $\mathbf{R}$ but $\{x_n\}$ need not be a Cauchy sequence in $\mathbf{R}$

(D) Neither $\{x_n\}$ nor $\{y_n\}$ needs to be a Cauchy sequence in $\mathbf{R}$

Q.6 Let $\vec{F} = 2xyz\,e^{x^2}\,\hat{i} + z\,e^{x^2}\,\hat{j} + y\,e^{x^2}\,\hat{k}$ be the gradient of a scalar function. The value of $\int_L \vec{F}\cdot d\vec{r}$ along the oriented path L from $(0, 0, 0)$ to $(1, 0, 2)$ and then to $(1, 1, 2)$ is

(A) 0 (B) $2e$ (C) e (D) e^2

Q.7 Let $\vec{F} = xy\,\hat{i} + y\,\hat{j} - yz\,\hat{k}$ denote the force field on a particle traversing the path L from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. The work done by $\vec{F}$ is

(A) 0 (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{2}$ (D) 1

Q.8 Let $\mathbf{R}[X]$ be the ring of real polynomials in the variable X. The number of ideals in the quotient ring $\mathbf{R}[X]/(X^2 - 3X + 2)$ is

(A) 2 (B) 3 (C) 4 (D) 6

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Q.9 Consider the differential equation $\frac{dy}{dx} = ay - by^2$, where $a, b > 0$ and $y(0) = y_0$. As $x \to +\infty$, the solution $y(x)$ tends to

(A) 0 (B) $\frac{a}{b}$ (C) $\frac{b}{a}$ (D) y_0

Q.10 Consider the differential equation $(x+y+1)\,dx + (2x+2y+1)\,dy = 0$. Which of the following statements is true?

(A) The differential equation is linear
(B) The differential equation is exact
(C) e^{x+y} is an integrating factor of the differential equation
(D) A suitable substitution transforms the differentiable equation to the variables separable form

Q.11 Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that $T((1,2)) = (2,3)$ and $T((0,1)) = (1,4)$. Then $T((5,6))$ is

(A) $(6,-1)$ (B) $(-6,1)$ (C) $(-1,6)$ (D) $(1,-6)$

Q.12 The number of 2×2 matrices over $\mathbf{Z}_3$ (the field with three elements) with determinant 1 is

(A) 24 (B) 60 (C) 20 (D) 30

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Q.13 The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^{n^2}$, where $a_0 = 1$, $a_n = 3^{-n} a_{n-1}$ for $n \in \mathbf{N}$, is

(A) 0 (B) $\sqrt{3}$ (C) 3 (D) ∞

Q.14 Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbf{R}^3$ is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Then T

(A) maps the subspace spanned by e_1 and e_2 into itself
(B) has distinct eigenvalues
(C) has eigenvectors that span $\mathbf{R}^3$
(D) has a non-zero null space

Q.15 Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard basis of $\mathbf{R}^3$ is $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, where a, b, c are real numbers not all zero. Then T

(A) is one-to-one
(B) is onto
(C) does not map any line through the origin onto itself
(D) has rank 1

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question No. Do not write anything else on this page.

Question No.	Answer	Do not write in this column
01		
02		
03		
04		
05		
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09		
10		
11		
12		
13		
14		
15		

FOR EVALUATION ONLY

No. of correct answers		Marks	(+)
No. of incorrect answers		Marks	(−)
Total marks in question nos. 1-15			( )

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Q.16 (a) Obtain the general solution of the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$

$$\frac{dy}{dt} = 4x - y + e^{3t}$$

(9)

(b) Find the curve passing through $\left(\frac{1}{2}, 0\right)$ and having slope at (x, y) given by the differential equation $2(1 + y^2)\, dx + (2x - \tan^{-1} y)\, dy = 0$. (6)

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Q.17 (a) Find the volume of the region in the first octant bounded by the sur[...] $x=0$, $y=x$, $y=2-x^2$, $z=0$ and $z=x^2$.

(b) Suppose $f:\mathbf{R}\to\mathbf{R}$ is a non-constant continuous function satisfying $f(x+y)=f(x)f(y)$ for all $x, y\in\mathbf{R}$.

(i) Show that $f(x)\neq 0$ for all $x\in\mathbf{R}$.

(ii) Show that $f(x)>0$ for all $x\in\mathbf{R}$.

(iii) Show that there exists $\beta\in\mathbf{R}$ such that $f(x)=\beta^x$ for all $x\in\mathbf{R}$. (9)

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Q.18 (a) Let $f(x)$ and $g(x)$ be real valued functions continuous in $[a,b]$, differentiab[le in] (a,b) and let $g'(x) \neq 0$ for all $x \in (a,b)$. Show that there exists $c \in (a,b)$ such tha[t] $\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$. (9)

(b) Let $0 < \lambda < 4$ and let $\{a_n\}$ be a sequence of positive real numbers satisfying $a_{n+1} = \lambda a_n^2(1-a_n)$ for $n \in \mathbf{N}$. Prove that $\lim_{n\to\infty} a_n$ exists and determine this limit. (6)

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Q.19 Let G be an open subset of $\mathbf{R}$.

(a) If $0 \notin G$, then show that $H = \{xy : x, y \in G\}$ is an open subset of $\mathbf{R}$. (9)

(b) If $0 \in G$ and if $x + y \in G$ for all $x, y \in G$, then show that $G = \mathbf{R}$. (6)

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Q.20 Let $p(x)$ be a non-constant polynomial with real coefficients such that $p(x) \neq 0$ fo
$x \in \mathbf{R}$. Define $f(x) = \dfrac{1}{p(x)}$ for all $x \in \mathbf{R}$. Prove that

(i) for each $\varepsilon > 0$, there exists $\alpha > 0$ such that $|f(x)| < \varepsilon$ for all $x \in \mathbf{R}$ satisfying $|x| > \alpha$, and

(ii) $f : \mathbf{R} \to \mathbf{R}$ is a uniformly continuous function. (15)

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Q.21 (a) Let $M(k)$ and $m(k)$ denote respectively the absolute maximum and the absolute minimum values of $x^3+9x^2-21x+k$ in the closed interval $[-10, 2]$. Find all the real values of k for which $|M(k)|=|m(k)|$. (6)

(b) Let $\alpha_1=0, \beta_1=1$; $\alpha_2=1, \beta_2=1$, and for $n \geq 3$,

$\alpha_n=\alpha_{n-1}+2\alpha_{n-2}$,

$\beta_n=\beta_{n-1}+2\beta_{n-2}$.

Prove that, for $n \in \mathbf{N}$

(i) $\beta_n=2\alpha_n+(-1)^{n-1}$

(ii) $\alpha_n+\beta_n=2^{n-1}$

Deduce that $\lim_{n\to\infty}\frac{\alpha_n a+\beta_n b}{2^{n-1}}=\frac{a+2b}{3}$ for any $a, b \in \mathbf{R}$. (9)

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Q.22 (a) Let $f(x,y)=\alpha x^2+xy+\beta y^2$, $\alpha\neq 0, \beta\neq 0, 4\alpha\beta\neq 1$. Find sufficient conditions on α, β such that $(0, 0)$ is

(i) a point of local maxima of $f(x, y)$

(ii) a point of local minima of $f(x, y)$

(iii) a saddle point of $f(x, y)$. (9)

(b) Find the derivative of $f(x, y, z)=7x^3-x^2z-z^2+28y$ at the point $A=(1,-1,0)$ along the unit vector $\frac{1}{7}(6\hat{i}-2\hat{j}+3\hat{k})$. What is the unit vector along which f decreases most rapidly at A? Also, find the rate of this decrease. (6)

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Q.23 Using $x = e^u$, transform the differential equation $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = \cos x$ to a second order differential equation with constant coefficients. Obtain the general solution of the transformed differential equation. (15)

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Q.24 Let G be a group and let $A(G)$ denote the set of all automorphisms of G, i.e., one-to-one, onto, group homomorphisms from G to G. An automorphism $f: G \to G$ of the form $f(x) = axa^{-1}, x \in G$ (for some $a \in G$) is called an inner automorphism. Let $I(G)$ denote the set of all inner automorphisms of G.

(a) Show that $A(G)$ is a group under composition of functions and that $I(G)$ is a normal subgroup of $A(G)$. (9)

(b) Show that $I(G)$ is isomorphic to $G/Z(G)$, where $Z(G) = \{g \in G : xg = gx \text{ for all } x \in G\}$ is the center of G. (6)

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Q.25 (a) Give an example of a linear transformation $T:\mathbf{R}^2 \to \mathbf{R}^2$ such that $T^2(v) =$ for all $v \in \mathbf{R}^2$. (6)

(b) Let V be a real n-dimensional vector space and let $T:V \to V$ be a linear transformation satisfying $T^2(v) = -v$ for all $v \in V$.

(i) Show that n is even.

(ii) Use T to make V into a complex vector space such that the multiplication by complex numbers extends the multiplication by real numbers.

(iii) Show that, with respect to the complex vector space structure on V obtained in (ii), $T:V \to V$ is a complex linear transformation. (9)

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Q.26 Let W be the region bounded by the planes $x = 0, y = 0, y = 3, z = 0$ and $x + 2z$ Let S be the boundary of this region. Using Gauss' divergence theorem, evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and $\hat{n}$ is the outward unit normal vector to S.

(15)

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Q.27 (a) Using Stokes' theorem evaluate the line integral $\int_L (y\hat{i} + z\hat{j} + x\hat{k}) \cdot \vec{dr}$, where L is the intersection of $x^2 + y^2 + z^2 = 1$ and $x + y = 0$ traversed in the clockwise direction when viewed from the point $(1, 1, 0)$. (9)

(b) Change the order of integration in the integral $\int_0^1 \int_{x-1}^{\sqrt{1-x^2}} f(x, y)\, dy\, dx$. (6)

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Q.28 In a group G, $x \in G$ is said to be conjugate to $y \in G$, written $x \sim y$, if there exists $z \in$ such that $x = zyz^{-1}$.

(a) Show that $\sim$ is an equivalence relation on G. Show that a subgroup N of G is a normal subgroup of G if and only if N is a union of equivalence classes of $\sim$. (6)

(b) Consider the group of all non-singular 3×3 real matrices under matrix multiplication. Show that $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (i.e., the two matrices are conjugate). (9)

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Q.29 Let S denote the commutative ring of all continuous real valued functions on $[0,1]$, un pointwise addition and multiplication. For $a \in [0,1]$, let $M_a = \{f \in S \mid f(a) = 0\}$.

(a) Show that M_a is an ideal in S. (6)

(b) Show that M_a is a maximal ideal in S. (9)

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