C

2009 – MA

Test Paper Code : MA

Time : 3 Hours Maximum Marks : 300

INSTRUCTIONS

1. The question-cum-answer booklet has **40** pages and has **29** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Roll Number, Name and the Name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer : (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **–2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone, pager and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on

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READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

ROLL NUMBER						

Name :

Test Centre :

Do not write your Roll Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

Signature of the Candidate

I have verified the information filled by the Candidate above.

Signature of the Invigilator

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Special Instructions/Useful Data

A^T	-	Transpose of the matrix A
C	-	Set of complex numbers
N	-	Set of natural numbers
Q	-	Set of rational numbers
R	-	Set of real numbers
Z	-	Set of integers

DO NOT WRITE ON THIS PAGE

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-29 (subjective questions) carry *fifteen* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 Let V be the vector space of all 6×6 real matrices over the field $\mathbb{R}$. Then the dimension of the subspace of V consisting of all symmetric matrices is

(A) 15 (B) 18 (C) 21 (D) 35

Q.2 Let R be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$ under point-wise addition and multiplication. Let $I = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a bounded function}\}$, $J = \{f : \mathbb{R} \to \mathbb{R} \mid f(3) = 0\}$. Then

(A) J is an ideal of R but I is not an ideal of R
(B) I is an ideal of R but J is not an ideal of R
(C) both I and J are ideals of R
(D) neither I nor J is an ideal of R

Q.3 Which of the following sequences of functions is uniformly convergent on $(0, 1)$?

(A) x^n (B) $\frac{n}{nx+1}$ (C) $\frac{x}{nx+1}$ (D) $\frac{1}{nx+1}$

Q.4 Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation satisfying $T^3 + 3T^2 = 4I$, where I is the identity transformation. Then the linear transformation $S = T^4 + 3T^3 - 4I$ is

(A) one-one but not onto (B) onto but not one-one
(C) invertible (D) non-invertible

Q.5 The number of all subgroups of the group $(Z_{60}, +)$ of integers modulo 60 is

(A) 2 (B) 10 (C) 12 (D) 60

C

Q.6 Let $a_n = \begin{cases} \frac{1}{3^n} & \text{if } n \text{ is a prime,} \\ \frac{1}{4^n} & \text{if } n \text{ is not a prime.} \end{cases}$

Then the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is

(A) 4 (B) 3 (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Q.7 The set of all limit points of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \ldots$ is

(A) $[0,1]$
(B) $(0,1]$
(C) the set of all rational numbers in $[0,1]$
(D) the set of all rational numbers in $[0,1]$ of the form $\frac{m}{2^n}$ where m and n are integers

Q.8 Let $F: \mathbb{R} \to \mathbb{R}$ be a continuous function and $a > 0$. Then the integral $\int_0^a \left[\int_0^x F(y)\, dy \right] dx$ equals

(A) $\int_0^a y F(y)\, dy$ (B) $\int_0^a (a-y) F(y)\, dy$

(C) $\int_0^a (y-a) F(y)\, dy$ (D) $\int_a^0 y F(y)\, dy$

Q.9 The set of all positive values of a for which the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \tan^{-1}\left(\frac{1}{n}\right) \right)^a$ converges, is

(A) $\left(0, \frac{1}{3}\right]$ (B) $\left(0, \frac{1}{3}\right)$ (C) $\left[\frac{1}{3}, \infty\right)$ (D) $\left(\frac{1}{3}, \infty\right)$

Q.10 Let a be a non-zero real number. Then $\lim_{x \to a} \frac{1}{x^2 - a^2} \int_a^x \sin(t^2)\, dt$ equals

(A) $\frac{1}{2a}\sin(a^2)$ (B) $\frac{1}{2a}\cos(a^2)$ (C) $-\frac{1}{2a}\sin(a^2)$ (D) $-\frac{1}{2a}\cos(a^2)$

Q.11 Let $T(x, y, z) = xy^2 + 2z - x^2z^2$ be the temperature at the point (x, y, z). The unit in the direction in which the temperature decreases most rapidly at $(1, 0, -1)$ is

(A) $-\frac{1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$

(B) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

(C) $\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}$

(D) $-\left(\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}\right)$

Q.12 Consider the differential equation $2\cos(y^2)\,dx - xy\sin(y^2)\,dy = 0$. Then

(A) e^x is an integrating factor

(B) e^{-x} is an integrating factor

(C) $3x$ is an integrating factor

(D) x^3 is an integrating factor

Q.13 Suppose $\vec{V} = p(x, y)\hat{i} + q(x, y)\hat{j}$ is a continuously differentiable vector field defined in a domain D in $\mathbb{R}^2$. Which one of the following statements is NOT equivalent to the remaining ones?

(A) There exists a function $\phi(x, y)$ such that $\frac{\partial \phi}{\partial x} = p(x, y)$ and $\frac{\partial \phi}{\partial y} = q(x, y)$ for all $(x, y) \in D$

(B) $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$ holds at all points of D

(C) $\oint_C \vec{V} \cdot d\vec{r} = 0$ for every piecewise smooth closed curve C in D

(D) The differential $p\,dx + q\,dy$ is exact in D

Q.14 Let $f, g: [-1, 1] \to \mathbb{R}$, $f(x) = x^3$, $g(x) = x^2|x|$. Then

(A) f and g are linearly independent on $[-1, 1]$

(B) f and g are linearly dependent on $[-1, 1]$

(C) $f(x)g'(x) - f'(x)g(x)$ is NOT identically zero on $[-1, 1]$

(D) there exist continuous functions $p(x)$ and $q(x)$ such that f and g satisfy $y'' + py' + qy = 0$ on $[-1, 1]$

Q.15 The value of c for which there exists a twice differentiable vector field $\vec{F}$ with $\text{curl}\,\vec{F} = 2x\hat{i} - 7y\hat{j} + cz\hat{k}$ is

(A) 0

(B) 2

(C) 5

(D) 7

Q.16 Container A contains 100 cc of milk and container B contains 100 cc of wate... liquid in A is transferred to B, the mixture is thoroughly stirred and 5 cc of th... in B is transferred back into A. Each such two-way transfer is called a dilution. Le... the percentage of water in container A after n such dilutions, with the understand... that $a_0 = 0$.

(a) Prove that $a_1 = \frac{100}{21}$ and that, in general, $a_n = \frac{100}{21} + \frac{19}{21} a_{n-1}$ for $n = 1, 2, 3, \ldots$. (6)

(b) Using (a) prove that $a_n = 50\left[1 - \left(\frac{19}{21}\right)^n\right]$ for $n = 1, 2, 3, \ldots$.

Find $\lim_{n\to\infty} a_n$ and explain why the answer is intuitively obvious. (9)

C

Q.17 (a) Let $f : \mathrm{N} \times \mathrm{N} \rightarrow \mathrm{R}$ be a non-negative function. Assume that for e the series $\sum_{n=1}^{\infty} f(m, n)$ is convergent and has sum a_m and further that the se

is also convergent and has sum L. Prove that for every n, the series $\sum_{m=1}^{\infty} f(m, n)$

convergent and if we denote its sum by b_n then the series $\sum_{n=1}^{\infty} b_n$ is also convergent and has sum L. (9)

(b) Define $f : \mathrm{N} \times \mathrm{N} \rightarrow \mathrm{R}$ by

$$f(m, n) = \begin{cases} 0 & \text{if } n > m, \\ \dfrac{-1}{2^{m-n}} & \text{if } n < m, \\ 1 & \text{if } n = m. \end{cases}$$

Show that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) = 2$ and $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m, n) = 0$. (6)

C

Q.18 (a) Evaluate $\iint_R \cos\left(\max\left\{x^3, y^{3/2}\right\}\right) dx\, dy$, where $R = [0,1] \times [0,1]$.

(b) Let $S = \sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{10000}$ and $I = \int_0^{10000} \sqrt{x}\, dx$. Show that $I \le S \le I + 10$

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Q.19 Let $D=\{(x, y) : x\geq 0, y\geq 0\}$. Let $f(x, y)=(x^2+y^2)e^{-x-y}$ for $(x, y)\in D$. Prove that f attains its maximum on D at two boundary points.

Deduce that $\dfrac{x^2+y^2}{4}\leq e^{x+y-2}$ for all $x\geq 0$, $y\geq 0$.

Q.20 (a) Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Show that the condition $a_2 b_1 > 0$ is suffic[illegible]t necessary for the system

$$\frac{dx}{dt} = a_1 x + b_1 y,$$
$$\frac{dy}{dt} = a_2 x + b_2 y,$$

to have two linearly independent solutions of the form $x = c_1 e^{\lambda_1 t}$, $y = d_1 e^{\lambda_1 t}$ and $x = c_2 e^{\lambda_2 t}$, $y = d_2 e^{\lambda_2 t}$ with $c_1, d_1, c_2, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$. (9)

(b) Show that the differential equation representing the family of all straight lines which have an intercept of constant length L between the coordinate axes is

$$x\frac{dy}{dx} - y = \frac{L\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$ (6)

C

Q.21 Let $A, B, k > 0$. Solve the initial value problem

$$\frac{dy}{dx} - Ay + By^3 = 0, \ x > 0, \ y(0) = k.$$

Also show that

(a) if $k < \sqrt{\frac{A}{B}}$, then the solution $y(x)$ is monotonically increasing on $(0, \infty)$ and tends to $\sqrt{\frac{A}{B}}$ as $x \to \infty$;

(b) if $k > \sqrt{\frac{A}{B}}$, then the solution $y(x)$ is monotonically decreasing on $(0, \infty)$ and tends to $\sqrt{\frac{A}{B}}$ as $x \to \infty$. (15)

C

Q.22 (a) Evaluate

$$\int_C \left(3y^2 + 2z^2\right) dx + (6x - 10z)y\, dy + \left(4xz - 5y^2\right) dz$$

along the portion from (1, 0, 1) to (3, 4, 5) of the curve C, which is the intersection of the two surfaces $z^2 = x^2 + y^2$ and $z = y + 1$. (6)

(b) A particle moves counterclockwise along the curve $3x^2 + y^2 = 3$ from (1, 0) to a point P, under the action of the force $\vec{F}(x, y) = \frac{x}{y}\hat{i} + \frac{y}{x}\hat{j}$. Prove that there are two possible locations of P such that the work done by $\vec{F}$ is 1. (9)

Q.23 Verify Stokes's theorem for the hemisphere $x^2+y^2+z^2=9,\ z\geq 0$ and the vect

$\vec{F}=(z^2-y)\hat{i}+(x-2yz)\hat{j}+(2xz-y^2)\hat{k}$.

C

Q.24 (a) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y, z) = (x+2y$ ...

Let $N(T)$ be the null space of T and $W = \left\{ \vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot \vec{u} = 0, \text{ for all } \vec{u} \in N(T) \right\}$. F...

a linear transformation $S: \mathbb{R}^2 \to W$ such that $TS = I$, where I is the identity transformation on $\mathbb{R}^2$. (9)

(b) Suppose A is a real square matrix of odd order such that $A + A^T = 0$. Prove that A is singular. (6)

Q.25 (a) Find all pairs (a, b) of real numbers for which the system of equations

$$x+3y=1$$
$$4x+ay+z=0$$
$$2x+3z=b$$

has (i) a unique solution, (ii) infinitely many solutions, (iii) no solution. (9)

(b) Let A be an $n \times n$ matrix such that $A^n = 0$ and $A^{n-1} \neq 0$. Show that there exists a vector $v \in \mathbb{R}^n$ such that $\{v, Av, \ldots, A^{n-1}v\}$ forms a basis for $\mathbb{R}^n$. (6)

C

Q.26 (a) In which of the following pairs are the two groups isomorphic to each other? Justify your answers.

(i) R/Z and S^1, where R is the additive group of real numbers and $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under complex multiplication.

(ii) (Z, +) and (Q, +). (9)

(b) Prove or disprove that if G is a finite abelian group of order n, and k is a positive integer which divides n, then G has at most one subgroup of order k. (6)

Q.27 Let I and J be ideals of a ring R. Let IJ be the set of all possible sums $\sum_{i=1}^{n} a_i$

$a_i \in I, b_i \in J$ for $i = 1, 2, \ldots, n$ and $n \in \mathrm{N}$.

(a) Prove that IJ is an ideal of R and $IJ \subseteq I \cap J$. (9)

(b) Is it true that $IJ = I \cap J$? Justify your answer. (6)

C

Q.28 A sequence $\{f_n\}$ of functions defined on an interval I is said to be uniformly b[illegible] on I if there exists some M such that $|f_n(x)| \le M$ for all $x \in I$ and for all $n \in \mathrm{N}$.

(a) Prove that if a sequence of functions $\{f_n\}$ converges to a function f on I and $\{f_n\}$ is uniformly bounded on I, then f is bounded on I. (6)

(b) Suppose the sequences $\{f_n\}$ and $\{g_n\}$ of functions converge uniformly to f and g respectively on I and both are uniformly bounded on I. Prove that the product sequence $\{f_n g_n\}$ converges to fg uniformly on I. Show by an example that this may fail if only one of $\{f_n\}$ and $\{g_n\}$ is uniformly bounded on I. (9)

Q.29 (a) Prove that if f is a real-valued function which is uniformly continuous interval (a, b), then f is bounded on (a, b).

(b) Let f be a differentiable function on an interval (a, b). Assume that f' is bounded on (a, b). Prove that f is uniformly continuous on (a, b). (6)