Coimisiún na Scrúduithe Stáit
State Examinations Commission

# LEAVING CERTIFICATE EXAMINATION, 2005

## MATHEMATICS – HIGHER LEVEL

**PAPER 1 ( 300 marks )**

**THURSDAY, 9 JUNE – MORNING, 9:30 to 12:00**

Attempt **SIX QUESTIONS** (50 marks each).

**WARNING:** **Marks will be lost if all necessary work is not clearly shown.**

**Answers should include the appropriate units of measurement, where relevant.**

**1.** **(a)** Solve the simultaneous equations:

$$\frac{x}{5}-\frac{y}{4}=0$$
$$3x+\frac{y}{2}=17.$$

**(b)** **(i)** Express $2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}+2^{\frac{1}{4}}$ in the form $2^{\frac{p}{q}}$, where $p, q \in \mathbf{Z}$.

**(ii)** Let $f(x)=ax^3+bx^2+cx+d$.
Show that $(x-t)$ is a factor of $f(x)-f(t)$.

**(c)** $(x-p)^2$ is a factor of $x^3+qx+r$.
Show that $27r^2+4q^3=0$.
Express the roots of $3x^2+q=0$ in terms of $p$.

**2.** **(a)** Solve for $x$: $|x-1|<7$, where $x \in \mathbf{R}$.

**(b)** The cubic equation $4x^3+10x^2-7x-3=0$ has one integer root and two irrational roots. Express the irrational roots in simplest surd form.

**(c)** Let $f(x)=\dfrac{x^2+k^2}{mx}$, where $k$ and $m$ are constants and $m \neq 0$.

**(i)** Show that $f(km)=f\left(\dfrac{k}{m}\right)$.

**(ii)** $a$ and $b$ are real numbers such that $a \neq 0, b \neq 0$ and $a \neq b$.
Show that if $f(a)=f(b)$, then $ab=k^2$.

**3.** **(a)** Given that $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, show that $A^3 = A^{-1}$.

**(b)** Solve the quadratic equation:

$$2iz^2 + (6+2i)z + (3-6i) = 0, \text{ where } i^2 = -1.$$

**(c)** **(i)** $z = \cos\theta + i\sin\theta$. Use De Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \text{ for } n \in \mathbf{N}.$$

**(ii)** Expand $\left(z + \frac{1}{z}\right)^4$ and hence express $\cos^4\theta$ in terms of $\cos 4\theta$ and $\cos 2\theta$.

**4.** **(a)** Write the recurring decimal 0.636363….. as an infinite geometric series and hence as a fraction.

**(b)** **(i)** The first three terms in the binomial expansion of $(1+kx)^n$ are $1 - 21x + 189x^2$.
Find the value of $n$ and the value of $k$.

**(ii)** A sequence is defined by $u_n = (2-n)2^{n-1}$.
Show that $u_{n+2} - 4u_{n+1} + 4u_n = 0$, for all $n \in \mathbf{N}$.

**(c)** **(i)** Show that

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}, \text{ where } a \text{ and } b \text{ are real numbers.}$$

**(ii)** The lengths of the sides of a right-angled triangle are $a$, $b$ and $c$, where $c$ is the length of the hypotenuse.
Using the result from part **(i)**, or otherwise, show that $a + b \leq c\sqrt{2}$.

**5.** **(a)** Solve for $x$: $\sqrt{10-x} = 4-x$.

**(b)** Prove by induction that

$$\sum_{r=1}^{n}(3r-2) = \frac{n}{2}(3n-1).$$

**(c)** **(i)** Show that $\dfrac{1}{\log_a b} = \log_b a$, where $a, b > 0$ and $a, b \neq 1$.

**(ii)** Show that

$$\frac{1}{\log_2 c} + \frac{1}{\log_3 c} + \frac{1}{\log_4 c} + \ \ldots\ldots\ldots + \frac{1}{\log_r c} = \frac{1}{\log_{r!} c}, \text{ where } c > 0, c \neq 1.$$

**6.** **(a)** Differentiate with respect to $x$

**(i)** $(1+7x)^3$ **(ii)** $\sin^{-1}\left(\dfrac{x}{5}\right)$.

**(b)** Let $y = \dfrac{1-\cos x}{1+\cos x}$.

Show that $\dfrac{dy}{dx} = t + t^3$, where $t = \tan\dfrac{x}{2}$.

**(c)** The equation of a curve is $y = \dfrac{x}{x-1}$, where $x \neq 1$.

**(i)** Show that the curve has no local maximum or local minimum point.

**(ii)** Write down the equations of the asymptotes and hence sketch the curve.

**(iii)** Show that the curve is its own image under the symmetry in the point of intersection of the asymptotes.

**7.** **(a)** Find from first principles the derivative of $x^2$ with respect to $x$.

**(b)** **(i)** The parametric equations of a curve are:

$$x = 8 + \ln t^2$$
$$y = \ln(2 + t^2), \text{ where } t > 0.$$

Find $\frac{dy}{dx}$ in terms of $t$ and calculate its value at $t = \sqrt{2}$.

**(ii)** Find the slope of the tangent to the curve $xy^2 + y = 6$ at the point (1, 2).

**(c)** **(i)** Write down a quadratic equation whose roots are $\pm\sqrt{k}$.

**(ii)** Hence use the Newton-Raphson method to show that the rule

$$u_{n+1} = \frac{(u_n)^2 + k}{2u_n}$$

can be used to find increasingly accurate approximations for $\sqrt{k}$.

**(iii)** Using the above rule and taking $\frac{3}{2}$ as the first approximation for $\sqrt{3}$, find the third approximation, as a fraction.

**8.** **(a)** Find **(i)** $\int (2 + x^3)dx$ **(ii)** $\int e^{3x}\, dx$.

**(b)** **(i)** Evaluate $\int_1^4 \frac{2x+1}{x^2 + x + 1} dx$.

**(ii)** Evaluate $\int_0^{\frac{\pi}{8}} \sin^2 2\theta \, d\theta$.

**(c)** **(i)** Evaluate $\int_1^2 \frac{1}{\sqrt{3 + 2x - x^2}} dx$.

**(ii)** Use integration methods to derive a formula for the volume of a cone.

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