Cayley 2018

C1. The positive integer N has six digits in increasing order. For example, 124 689 is such a number.

However, unlike 124 689, three of the digits of N are 3, 4 and 5, and N is a multiple of 6.

How many possible six-digit integers N are there?

Solution

The digits of N increase, so the known, consecutive digits 3, 4 and 5 must be adjacent and in increasing order.

Since N has six digits, and they increase, the final digit must be at least 5.

Since N is a multiple of 6, it is even, so the final digit is even and must be 6 or 8. This means that 3 (as the start of 345) must be one of the first three digits of N.

This gives the following possible values of N:

345678, 134568, 134578, 234568, 234578, 123456 and 123458

Since N is a multiple of 3, the sum of the digits of N must be a multiple of 3.

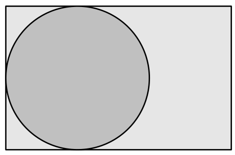
The possibilities 134578, 234568, 234578 and 123458 have digit sums 28, 28, 29 and 23 respectively so cannot be values for N.

But 345678, 134568, 123456 have digit sums 33, 27 and 21 respectively.

Therefore these are the three possible six-digit integers N.

[An alternative interpretation of the question allows digits in the solutions to repeat, since such a number would have digits that are in increasing order, even though the digits themselves do not always increase. This interpretation yields 19 solutions.]

C2. A circle lies within a rectangle and touches three of its edges, as shown.



The area inside the circle equals the area inside the rectangle but outside the circle.

What is the ratio of the length of the rectangle to its width?

Solution

Let the radius of the circle be r and the length of the rectangle be s.

Therefore the width of the rectangle is $2r$.

The area inside the circle is πr^2.

The area inside the rectangle but outside the circle is $2rs - \pi r^2$.

These quantities are equal so

$$\pi r^2 = 2rs - \pi r^2.$$

Adding πr^2 to each side gives

$$2\pi r^2 = 2rs.$$

Since r is not zero, division by $2r$ is permissible and gives

$$\pi r = s.$$

Therefore the ratio of the length of the rectangle to the width is $\pi r : 2r$, which simplifies to $\pi : 2$.

Cayley 2018

C3. The addition sum $XCV + XXV = CXX$ is true in Roman numerals.

In this question, however, the sum is actually the letter-sum shown alongside, in which: each letter stands for one of the digits 0 to 9, and stands for the same digit each time it occurs; different letters stand for different digits; and no number starts with a zero.

$$\begin{array}{r} X\,C\,V \\ +\,X\,X\,V \\ \hline C\,X\,X \end{array}$$

Find all solutions, and explain how you can be sure you have found every solution.

Solution

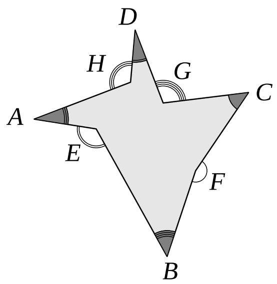
First, since C is the leading digit of a three-digit number, $C > 0$.

From the tens column we see that $C + X$ results in an X. Since C cannot be 0, there must be a carry digit of 1 from the units column, and $C + 1 = 10$. Therefore $C = 9$ and V is at least 5.

Now from the hundreds column $X = 4$, so from the units column $V = 7$.

Thus the unique solution is $497 + 447 = 944$.

C4. Prove that the difference between the sum of the four marked interior angles A, B, C, D and the sum of the four marked exterior angles E, F, G, H of the polygon shown is 360°.



Solution

Let the sum of the four marked interior angles be S.

The four remaining interior angles are $360 - E$, $360 - F$, $360 - G$ and $360 - H$ since the angles at a point add up to 360 degrees.

The polygon has eight sides so its interior angles add up to 1080 degrees. Hence $1080 = S + 360 - E + 360 - F + 360 - G + 360 - H$.

This equation simplifies to $1080 = 1440 + S - (E + F + G + H)$, which is equivalent to $E + F + G + H - S = 360$.

Hence the difference between the sum of the four marked interior angles and the four marked exterior angles is 360 degrees, as required.

C5. In the expression below, three of the + signs are changed into − signs so that the expression is equal to 100:

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$+ 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

In how many ways can this be done?

Solution

The given expression is equal to 210.

Changing three signs from + to – will decrease the expression by double the sum of the three affected terms. So the sum of the three terms must be half of 210 – 100, which is 55.

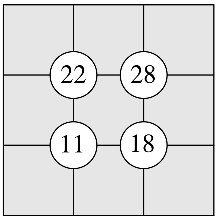
The three terms must be distinct integers that are less than or equal to 20. The largest possible total for three such terms is $18 + 19 + 20$, which is 57. To achieve a total of 55, we need to decrease the sum of the three terms by 2. This can be done by reducing the 18 by 2 or reducing the 19 by 2 (which is equivalent to taking one off each of the 18 and 19).

Hence, the expression can be changed to give a total of 100 in two ways – either by changing the signs on the 16, 19 and 20 or on the 17, 18 and 20.

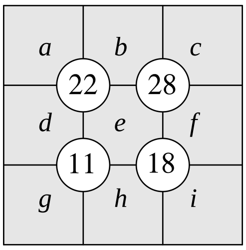
Cayley 2018

C6. In the puzzle *Suko*, the numbers from 1 to 9 are to be placed in the spaces (one number in each) so that the number in each circle is equal to the sum of the numbers in the four surrounding spaces.

How many solutions are there to the *Suko* puzzle shown alongside?



Solution



Using the labelling shown in the diagram, $d + e + g + h = 11$ where d, e, g and h are distinct positive integers. The smallest four positive distinct integers (1, 2, 3 and 4) have total 10. To give a total of 11, the only integer that can increase by one and remain distinct is the 4, which means that d, e, g and h must be 1, 2, 3 and 5, in some order.

Similarly, $b + c + e + f = 28$, which means that b, c, e and f must be 5, 6, 8 and 9, in some order, or 4, 7, 8 and 9 in some order.

Since e is the only variable in both lists and 5 is the only number in both lists, e must be 5, and b, c and f must be 6, 8 and 9 in some order.

The only numbers from 1 to 9 not accounted for so far are 4 and 7 so these must be the values of a and i, in some order.

If $a = 4$ and $i = 7$, consider the value of f. Since b, c, e and f must be 5, 6, 8 and 9, in some order, and $e \neq f$, f must be at least 6.

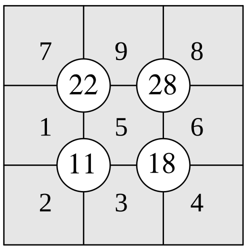
But $e + f + h + i = 5 + f + h + 7 = 18$, which would make h less than or equal to 0, which is impossible.

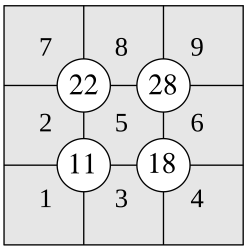
Hence $a = 7$ and $i = 4$.

Consider the value of f. If $f = 9$, $e + f + h + i = 5 + 9 + h + 4 = 18$, so $h = 0$, which is impossible.

If $f = 8$, it follows that $h = 1$ and $d = 2$ or 3, and hence $g = 3$ or 2 respectively. This would mean $b = 8$ or 7 respectively, both of which are impossible.

If $f = 6$, it follows that $h = 3$ and $d = 1$ or 2, and hence $g = 2$ or 1 respectively. This would mean $b = 9$ or 8, respectively. These values are possible and give the complete solutions:





There are two solutions.

Hamilton 2018

H1. The positive integers m and n satisfy the equation $20m + 18n = 2018$.
How many possible values of m are there?

Comment

There are many approaches to this question. It is straightforward to show that, given a solution $m = M$ and $n = N$, the pair $m = (M - 9k)$ and $n = (N + 10k)$ is also a solution for any integer value of k. This allows one to 'spot' a solution and generate a set of solutions from that one.

However, if we use this approach, it is important to show that there aren't any other solutions which are not in this set. (Indeed, in any question where we are trying to find all possible solutions, finding them is only half the battle − a key step is to show that we have found them *all*.)

The first of these solutions shows a way to complete the argument introduced above; the second demonstrates a slightly different approach.

Solution 1

Let M and N be a solution to the given equation, so $20M + 18N = 2018$.

For any integer k, the pair $m = (M - 9k)$ and $n = (N + 10k)$ is also a solution, since

$$20(M - 9k) + 18(N + 10k) = 20M - 180k + 18N + 180k = 2018.$$

It is easy to verify (and not too difficult to spot) that $m = 100$, $n = 1$ is a solution.

This quickly leads us to the following twelve solutions, which are all of the form $m = 100 - 9k, n = 1 + 10k$ for $k = 0, 1, 2, \dots, 11$:

m	100	91	82	73	64	55	46	37	28	19	10	1
n	1	11	21	31	41	51	61	71	81	91	101	111

We will now show that *any* solution to the given equation *must* be of the form $m = 100 - 9k, n = 1 + 10k$.

Let (M, N) be a pair of integers for which $20M + 18N = 2018$.

Since $20 \times 100 + 18 \times 1 = 2018$, we can subtract to give

$$20(100 - M) + 18(1 - N) = 0.$$

Then

$$10(100 - M) = 9(N - 1). \qquad (*)$$

Since 10 and 9 have no common factor greater than 1, $(100 - M)$ must be a multiple of 9, so we can write $100 - M = 9k$ for some integer k. This leads to $M = 100 - 9k$.

Then we have, from $(*)$, $90k = 9(N - 1)$, which leads to $N = 1 + 10k$.

So we have now shown that any pair of integers (M, N) which are a solution to the equation must be of the form $(100 - 9k, 1 + 10k)$ for some integer k.

For $100 - 9k$ to be positive we need $k \leqslant 11$ and for $1 + 10k$ to be positive we need $k \geqslant 0$, so the twelve solutions listed above are the only twelve.

Solution 2

If $20m + 18n = 2018$ then

$$\begin{aligned} 10m + 9n &= 1009 \\ 10m &= 1009 - 9n \\ 10m &= 1010 - (9n + 1) \\ m &= 101 - \frac{9n + 1}{10}. \end{aligned}$$

Since m is an integer, $9n + 1$ must be a multiple of 10, which means that $9n$ must be one less than a multiple of 10. For this to be true, the final digit of $9n$ must be 9, which happens exactly when the final digit of n is 1.

Every value of n with final digit 1 will produce a corresponding value of m. No other value of n is possible (and hence no other possible values of m possible either).

So now we just need to count the number of positive values of n with final digit 1 which also give a positive value of m.

Since m is positive, $\frac{9n + 1}{10}$ can be at most 100, with equality when $n = 111$.

Hence $0 < n \leqslant 111$. There are 12 values of n in this range which have final digit 1, and so there are 12 possible values of m.

[*Note that we have not found any of the solutions to the given equation; we did not need to do this in order to answer the question.*]

Hamilton 2018

H2. How many nine-digit integers of the form '*pqrpqrpqr*' are multiples of 24?
(*Note that p, q and r need not be different*.)

Comment

When attempting to solve questions like this one, it is natural to turn to divisibility tests (which are often learned right at the start of secondary school). One key property of these tests is that they work in both directions: for example, all numbers which are divisible by 8 have their final three digits divisible by 8, and conversely all numbers with final three digits divisible by 8 are themselves divisible by 8. In the first solution below, this property is essential – without it we would not be able to guarantee that we have found all results.

Solution 1

In order to be a multiple of 24, any number must be both a multiple of 3 and a multiple of 8. Conversely, since 8 and 3 are *coprime* (i.e. they have no common factor greater than 1), any number which is a multiple of both 3 and 8 must be a multiple of 24. It is therefore sufficient for us to count all the numbers of the required form which are multiples of both 3 and 8.

The digit sum of '*pqrpqrpqr*' is $3(p + q + r)$, which is always a multiple of 3, so all numbers of the form '*pqrpqrpqr*' are divisible by 3 (by the divisibility test for 3).

So now all we need to do is to count all the numbers of the required form which are multiples of 8, since they will also be a multiple of 3 and hence a multiple of 24.

By the divisibility test for 8, we need the three-digit number '*pqr*' to be divisible by 8, and each three-digit multiple of 8 will give us a nine-digit number divisible by 8. So all that remains is to count the three-digit multiples of 8.

The smallest of these is $104 (= 13 \times 8)$ and the largest is $992 (= 124 \times 8)$, so there are 112 three-digit multiples of 8 and therefore 112 numbers of the form '*pqrpqrpqr*' which are multiples of 24.

Solution 2

As in Solution 1, we can establish that we need '*pqrpqrpqr*' to be a multiple of both 3 and 8.

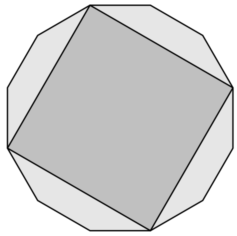
Notice that '*pqrpqrpqr*' $= 1001001 \times pqr$. 1001001 is divisible by 3 so every number of the form is '*pqrpqrpqr*' divisible by 3. Since 1001001 is odd, we need the three-digit number '*pqr*' to be divisible by 8.

There are 124 multiples of 8 less than 1000; of these 12 are less than 100, so there are 112 three-digit multiples of 8 and hence 112 numbers with the properties given in the question.

Hamilton 2018

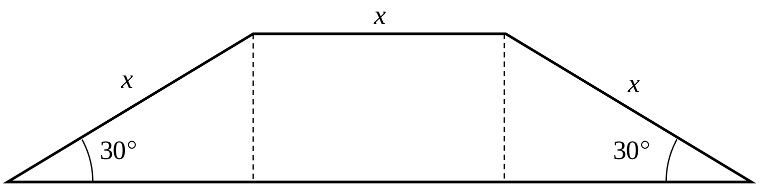
H3. The diagram shows a regular dodecagon and a square, whose vertices are also vertices of the dodecagon.

What is the value of the ratio
area of the square : area of the dodecagon?



Comment
There are several ways to attempt this question. If you know the formula $\frac{1}{2}ab\sin C$ for the area of a triangle, the result can be arrived at very quickly (see Solution 2). The majority of candidates offered a method similar to that described in Solution 1.

Solution 1
Let the side length of the dodecagon be x. Consider one of the four trapezia created by the edge of the square and the edges of the dodecagon.
The interior angle of a dodecagon is 150°, which quickly leads to the fact that the angles at the base of the trapezium are both equal to 30°.
We can then split the trapezium up into a rectangle and two right-angled triangles:

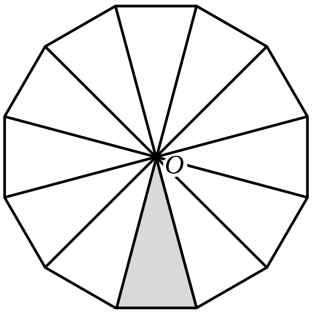


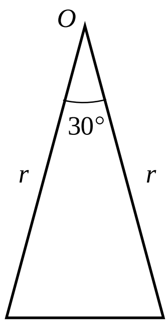
Then, by recognising that each of the triangles in the trapezium is half an equilateral triangle (or by using the well-known values of sin 30° and cos 30°), we can establish that the height of this trapezium is $x/2$ and the base is $x + x\sqrt{3}$.
The area of the square is therefore $\left(x + x\sqrt{3}\right)^2 = 2x^2\left(2 + \sqrt{3}\right)$ and the area of the dodecagon is equal to (the area of the square) + (4 × the area of the trapezium) which is

$$2x^2\left(2 + \sqrt{3}\right) + 4 \times \frac{x + \left(x + x\sqrt{3}\right)}{2} \times \frac{x}{2} = 3x^2\left(2 + \sqrt{3}\right).$$

We can divide through by the common factor of $x^2\left(2 + \sqrt{3}\right)$ to show that the required ratio is 2: 3.

Solution 2
Since the dodecagon is regular, all its diagonals intersect at the same point (its centre). Call this point O. Since the diagonals of the square are two of the diagonals of the dodecagon, they intersect at O. Hence O is also the centre of the square.
Let each diagonal have length $2r$.
Then the dodecagon comprises 12 congruent isosceles triangles radiating from O, each with two sides equal to r and vertex angle 30°:





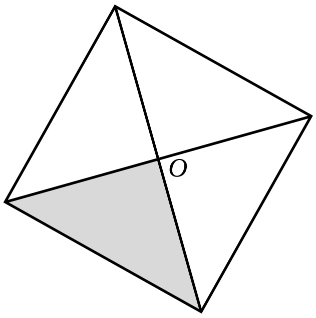
The area of the dodecagon is equal to 12 × the area of one of these triangles, which is equal to

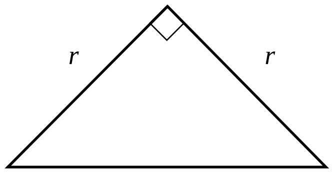
$$12 \times \left[\tfrac{1}{2} \times r \times r \times \sin 30^\circ\right].$$

Since $\sin 30^\circ = \frac{1}{2}$, this simplifies to

$$\text{Area of dodecagon} = 3r^2.$$

The square can be split into four congruent isosceles right-angled triangles, with perpendicular sides of length r.





The area of the square is equal to 4 × the area of one of these triangles, which is equal to

$$4 \times \left[\tfrac{1}{2} \times r \times r\right],$$

which simplifies to

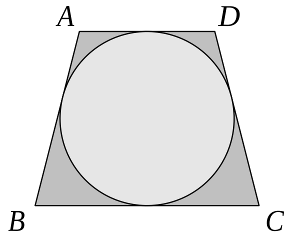
$$\text{Area of square} = 2r^2.$$

The required ratio is therefore $2r^2 : 3r^2$ which is equal to **2 : 3.**

Hamilton 2018

H4. The diagram shows a circle and a trapezium *ABCD* in which *AD* is parallel to *BC* and $AB = DC$. All four sides of *ABCD* are tangents of the circle. The circle has radius 4 and the area of *ABCD* is 72.

What is the length of *AB*?



Comment

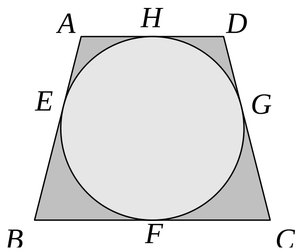
The solution to this question is dependent on the circle theorem which states that the two tangents to a circle from any external point are equal in length. The proof of this is straightforward (join the point to the centre of the circle and find two triangles which are congruent) − if you haven't seen it before you may want to work it through to satisfy yourself that it is indeed true.

Solution

Note that the height of the trapezium is equal to the diameter of the circle = 8.

Since the area of the trapezium is 72, we have $\frac{AD + BC}{2} \times 8 = 72$ and so $AD + BC = 18$.

Let *E*, *F*, *G* and *H* be the points on *AB*, *BC*, *CD* and *DA* respectively where the circle is tangent to the sides of the trapezium, as shown in the diagram below.



Since the two tangents to a circle from any external point are equal in length, we know that $AE = AH$.

Similarly, $BE = BF$, $CF = CG$ and $DG = DH$.

Then $AB + DC = (AE + BE) + (CG + DG) = AH + BF + CF + DH = AD + BC$, which we know is equal to 18.

Since $AB = DC$, we have $AB + DC = 2AB = 18$ and hence $AB = 9$.

H5. A two-digit number is divided by the sum of its digits. The result is a number between 2.6 and 2.7.

Find all of the possible values of the original two-digit number.

Solution

Let the number be 'ab' (where $a \neq 0$), which can be written as $10a + b$.

The value we are concerned with is $\dfrac{10a + b}{a + b}$.

We are given that $2.6 < \dfrac{10a + b}{a + b} < 2.7$.

Since $a + b$ must be positive (since both a and b are positive), we can multiply through by $(a + b)$ without changing the direction of the inequality signs, to give:

$$2.6\,(a + b) < 10a + b < 2.7\,(a + b).$$

The first of these two inequalities leads to

$$1.6b < 7.4a,$$

which can be rearranged to give

$$b < \frac{37}{8}a = 4\tfrac{5}{8}a. \qquad (1)$$

The second inequality leads to

$$7.3a < 1.7b,$$

which can be rearranged to give

$$b > \frac{73}{17}a = 4\tfrac{5}{17}a. \qquad (2)$$

From (2), we can deduce that $b > 4a$, so (since b is a single digit) a can be at most 2.

If $a = 1$, combining (1) and (2) gives $4\tfrac{5}{17} < b < 4\tfrac{5}{8}$, which is impossible (since b is a digit and hence an integer).

If $a = 2$, combining (1) and (2) gives $8\tfrac{10}{17} < b < 9\tfrac{1}{4}$, which gives $b = 9$.

These values of a and b give the number 29.

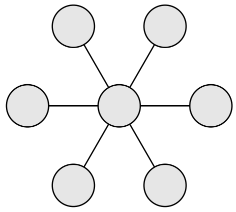
We must check that 29 does have the property we are looking for, which we can do by dividing directly:

$$\frac{29}{11} = 2\frac{7}{11} = 2.6363\ldots\ ,$$

which is indeed between 2.6 and 2.7, so the only possible value of the two-digit number is 29.

Hamilton 2018

H6. The figure shows seven circles joined by three straight lines.



The numbers 9, 12, 18, 24, 36, 48 and 96 are to be placed into the circles, one in each, so that the product of the three numbers on each of the three lines is the same.

Which of the numbers could go in the centre?

Comment

It is possible to argue that, once the centre number has been selected, the other six must be paired 'largest with smallest' etc. – but a complete convincing argument can be difficult to assemble. The method below demonstrates an alternative way (which was chosen by the majority of successful candidates) to show the only possible solutions.

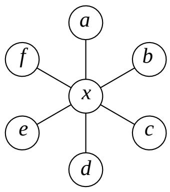
It is worth remembering that showing that 12 and 96 are the only centre numbers that could possibly work does not fully answer the question – we must show that arrangements are indeed possible (and this is easily done by direct construction).

Solution

Since we are interested in the products of various numbers, it seems natural to find the prime factorisations of the seven numbers we are given:

Number	9	12	18	24	36	48	96
Prime Factorisation	3^2	$2^2 \times 3$	2×3^2	$2^3 \times 3$	$2^2 \times 3^2$	$2^4 \times 3$	$2^5 \times 3$

Let the six numbers around the edge of the figure be a, b, c, d, e and f, and the central number x (as in the diagram below).



Then we need $a \times x \times d = b \times x \times e = c \times x \times f$. Since $x \neq 0$, we can divide by x, giving $ad = be = cf$.

Now consider the number $abcdef$. This must be a cube (since it is equal to, for example, $(ad)^3$). This number can also be written as $\frac{abcdefx}{x}$, which is useful since $abcdefx$ is just the product of the seven numbers we have been given. Hence $abcdef = \frac{2^{17} \times 3^{10}}{x}$, and, if we write x as $2^m \times 3^n$, we have $abcdef = 2^{17-m} \times 3^{10-n}$.

Note that, since x must be one of the given numbers, there are only a few possible values for m and n; in particular $m \leqslant 5$ and $n \leqslant 2$.

Since $abcdef$ is a cube, 2^{17-m} and 3^{10-n} must both be cubes too, which means that both $17 - m$ and $10 - n$ must be multiples of 3. Hence m could be equal to 2 or 5, and n must be equal to 1.

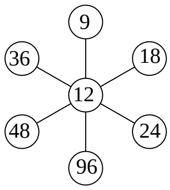
Hence $(m, n) = (2, 1)$ which gives $x = 12$, or $(m, n) = (5, 1)$ which gives $x = 96$.

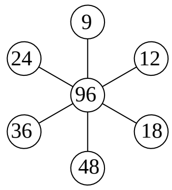
As mentioned in the comment above, we must now check that each of these cases enables the shape to be filled as required by the question:

When $x = 12$, we have $abcdef = 2^{15} \times 3^9$, so each pair of outside numbers must multiply to $2^5 \times 3^3$.

When $x = 96$, we have $abcdef = 2^{12} \times 3^9$, so each pair of outside numbers must multiply to $2^4 \times 3^3$.

In each case, a small amount of experimentation quickly leads to arrangements that work in each case, such as:





Hence both cases are possible and there are two numbers that could go in the centre space; namely 12 and 96.

Maclaurin 2018

M1. The sum of the squares of two real numbers is equal to fifteen times their sum. The difference of the squares of the same two numbers is equal to three times their difference.

Find all possible pairs of numbers that satisfy the above criteria.

Solution

Let the numbers be a and b. Then $a^2 + b^2 = 15(a + b)$ and $a^2 - b^2 = \pm 3(a - b)$.

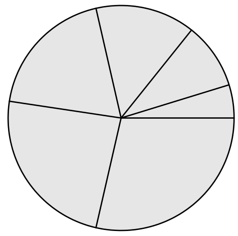
We do not know whether a and b are positive or negative, so if, for instance, $a = 4$ and $b = 3$, the expressions $a^2 - b^2$ and $a - b$ are both positive, but if $a = -4$ and $b = 3$ the first is positive and the second is negative. The ± sign is to take care of all alternatives. In fact, we have $a^2 + b^2 \geqslant 0$ in the first equation, so we know that $a + b \geqslant 0$. Now, by difference of two squares, we can write $a^2 - b^2$ as $(a - b)(a + b)$ in the second equation. Hence, if $a - b \neq 0$, we can cancel this term to obtain $a + b = \pm 3$, and so we know that the positive sign is appropriate.

We now have $a + b = 3$ and $a^2 + b^2 = 45$. Writing $b = 3 - a$, we obtain the quadratic $a^2 - 3a - 18 = 0$ and solutions $(6, -3)$ and $(-3, 6)$, so the numbers are 6 and -3 in either order.

However, there is also the possibility that $a - b = 0$, but if that is the case the first equation becomes $2a^2 = 30a$ and so $a = 0$ or 15. Hence the pair of numbers is either 0 and 0, 15 and 15 or 6 and -3.

Maclaurin 2018

M2. The diagram shows a circle that has been divided into six sectors of different sizes.

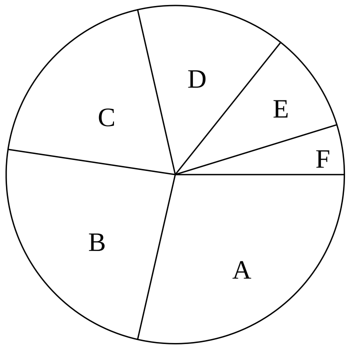


Two of the sectors are to be painted red, two of them are to be painted blue, and two of them are to be painted yellow. Any two sectors which share an edge are to be painted in different colours.

In how many ways can the circle be painted?

Solution

Label the sectors as in the diagram.



Suppose A is allocated a colour, which we call X. This can be done in three ways. Then neither B nor F can be coloured with X.

- If C is coloured with X, then B can be allocated a second colour in two ways. E must have the same colour as B since otherwise either D and E or E and F would have the same colour. Hence F and D have the third colour. This leads to 6 colourings.
- Similarly, if E is coloured with X, then F and C share a colour and B and D share the third. Again there are 6 colourings.
- If D is coloured with X, then E and F have different colours and, independently, so do B and C. This leads to a further 12 colourings.

The total number of colourings is 24.

Maclaurin 2018

M3. Three positive integers have sum 25 and product 360.
Find all possible triples of these integers.

Solution

Let the integers be a, b, c. Then $a + b + c = 25$ and $abc = 360 = 2^3 \times 3^2 \times 5$. We must split this factorised product up into parts so that the sum of the resulting numbers is 25.

It is possible, of course, to make a list of all the different ways of doing this, but there are 32 possibilities and ensuring that no alternatives are missing is difficult. Such an approach depends on the list being exhaustive and so omissions result in it being invalid. However, there are several methods which avoid making such a list.

Method 1:

This focuses on the fact that one of the three numbers is divisible by 5. Without loss of generality, take this to be a. It is either 5, 10, 15 or 20, since 25 is too large to obtain the required sum.

If $a = 5$, then $b + c = 20$ and $bc = 72$. It is possible to list all the factorisations of 72 and show that none of them produce a sum of 20. However, a better approach is to consider the quadratic $x^2 - 20x + 72 = 0$. The sum of its roots is 20 and the product is 72, so the roots will be b and c. However, the discriminant of this quadratic is negative, so there are no real roots.

This approach is useful in checking all such cases.

If $a = 10$, the quadratic is $x^2 - 15x + 36 = 0$, which has roots 3 and 12. Hence there is a solution 10, 3, 12.

If $a = 15$, the quadratic is $x^2 - 10x + 24 = 0$, which has roots 4 and 6, and so there is a solution 15, 4, 6.

If $a = 20$, the quadratic is $x^2 - 15x + 18 = 0$, and again the discriminant is negative and there are no roots.

Hence there are exactly two triples of numbers, namely 4, 6, 15 and 3, 10, 12.

Method 2:

This focuses on the largest number of the three. Since their sum is 25, this is greater than 8 and smaller than 24. Since it is also a factor of 360, the only possible values are 9, 10, 12, 15, 18 and 20. The approach in Method 1 can now be used to eliminate 9, 18 and 20. The values 10 and 12 result in the same triple.

Method 3:

This focuses on the 2s in the factorisation of 360. Since $a + b + c = 25$, they cannot be all even, nor can two be odd and one even, and since $abc = 360$, they cannot be all odd. It follows that two are even and one is odd. The factor of 8 is split between the two even numbers.

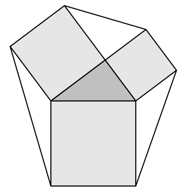
Without loss of generality, let $a = 2a_1$ and $b = 4b_1$, with a_1, b_1, c formed from the factors 3, 3 and 5. Now b_1 is not a multiple of 5 or 9, since the sum would be greater than 25, so it is either 1 or 3.

If $b_1 = 1$, then $b = 4$ and $2a_1 + c = 21$. Using the factors 3, 3 and 5, the only possibility is $a_1 = 3$, $c = 15$ and so $a = 6$.

If $b_1 = 3$, then $b = 12$ and $2a_1 + c = 13$. Using the factors 3 and 5, the only possibility is $a_1 = 5$, $c = 3$ and so $a = 10$.

Hence there are two triples of numbers, namely 4, 6, 15 and 3, 10, 12.

M4. The squares on each side of a right-angled scalene triangle are constructed and three further line segments drawn from the corners of the squares to create a hexagon, as shown. The squares on these three further line segments are then constructed (outside the hexagon).

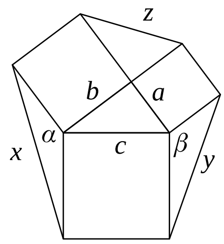


The combined area of the two equal-sized squares is 2018 cm^2.

What is the total area of the six squares?

Solution

Label the lengths as in the diagram below. Note that $z = c$ since it is the hypotenuse of a right-angled triangle with sides a and b. By chasing angles $\alpha = \pi - A$ and $\beta = \pi - B$. Hence $\cos\alpha = -\frac{a}{c}$ and $\cos\beta = -\frac{b}{c}$. We are told that $2c^2 = 2018$ so $c^2 = 1009$.



We have $x^2 = b^2 + c^2 - 2bc\cos\alpha = 3b^2 + c^2$ by the cosine rule, and similarly $y^2 = 3a^2 + c^2$.

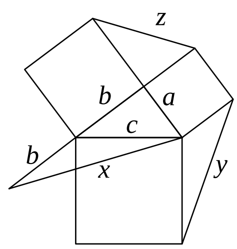
Now the sum of the six squares is

$$a^2 + b^2 + c^2 + \left(3b^2 + c^2\right) + \left(3a^2 + c^2\right) + c^2 = 4\left(a^2 + b^2\right) + 4c^2.$$

By Pythagoras, $a^2 + b^2 = c^2$ so the sum is $8c^2$, which is 8072.

Note that the cosine rule can be avoided by the construction shown. This produces a right-angled triangle whose sides are a and $2b$ whose hypotenuse is x.

It follows that $x^2 = a^2 + 4b^2$ and $y^2 = b^2 + 4a^2$.



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M5. For which integers n is $\dfrac{16\left(n^2 - n - 1\right)^2}{2n - 1}$ also an integer?

Solution

We set $m = 2n - 1$, so $n = \frac{1}{2}(m + 1)$. Now the expression becomes

$$\frac{16}{m}\left[\left(\frac{m + 1}{2}\right)^2 - \left(\frac{m + 1}{2}\right) - 1\right]^2 = \frac{1}{m}\left[(m + 1)^2 - 2(m + 1) - 4\right]^2$$

$$= \frac{1}{m}\left[m^2 - 5\right]^2 = \frac{m^4 - 10m^2 + 25}{m}.$$

This is an integer if, and only if, m is a factor of 25. Hence the values of m are $\pm 1,\ \pm 5,\ \pm 25$ and the corresponding values of n are $-12,\ -2,\ 0,\ 1,\ 3$ and 13.

Instead of using the substitution, the numerator can be rewritten as $\left((2n - 1)^2 - 5\right)^2$ to obtain the same result.

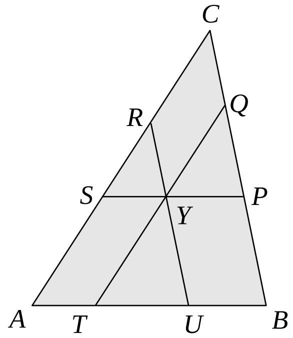
Maclaurin 2018

M6. The diagram shows a triangle ABC and points T, U on the edge AB, points P, Q on BC, and R, S on CA, where:

(i) SP and AB are parallel, UR and BC are parallel, and QT and CA are parallel;

(ii) SP, UR and QT all pass through a point Y; and

(iii) $PQ = RS = TU$.

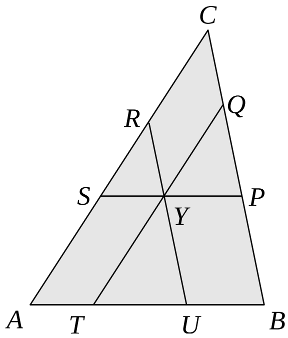


Prove that

$$\frac{1}{PQ} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$

Solution

Let $TU = PQ = RS = k$, $AB = c$, $BC = a$ and $CA = b$.



Triangles $\triangle RSY$ and $\triangle CAB$ are similar, since RS is parallel to CA, SY is parallel to AB and YR is parallel to BC, so $SY = \frac{ck}{b}$.

Similarly $YP = \frac{ck}{a}$. Now

$$\begin{aligned} c &= AT + k + UB = SY + k + YP \\ &= \frac{ck}{b} + k + \frac{ck}{a} = \frac{ab + bc + ca}{ab}k \end{aligned}$$

and so $\frac{1}{c} = \frac{ab}{(ab + bc + ca)k}$.

If three such expressions are added, the factor $ab + bc + ca$ cancels and we obtain the desired result.