Solutions to the Olympiad Maclaurin Paper

M1. The positive integer N has five digits.
The six-digit integer P is formed by appending the digit 2 to the front of N.
The six-digit integer Q is formed by appending the digit 2 to the end of N.

Given that $Q = 3P$, what values of N are possible?

Solution

We have $P = N + 200\,000$ and $Q = 10N + 2$. Hence

$$\begin{aligned} 10N + 2 &= 3 \times (N + 200\,000) \\ &= 3N + 600\,000. \end{aligned}$$

Therefore, subtracting $3N + 2$ from each side, we get

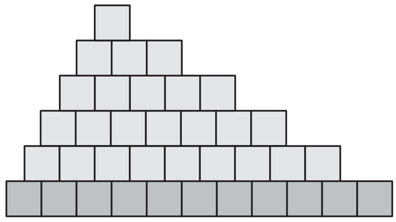
$$7N = 599\,998$$

and so, dividing each side by 7, we obtain

$$N = 85\,714.$$

Hence the only possible value of N is 85 714.

M2. A 'stepped' shape, such as the example shown, is made from 1×1 squares in the following way.



(i) There are no gaps or overlaps.

(ii) There are an odd number of squares in the bottom row (eleven in the example shown).

(iii) In every row apart from the bottom one, there are two fewer squares than in the row immediately below.

(iv) In every row apart from the bottom one, each square touches two squares in the row immediately below.

(v) There is one square in the top row.

Prove that $36A = (P + 2)^2$, where A is the area of the shape and P is the length of its perimeter.

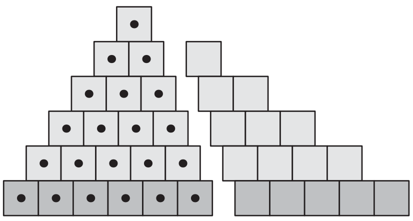
Solution

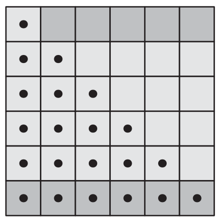
Let n be the number of steps in the shape.

Now the top row consists of one square, and each subsequent row has two more squares than the one before. Hence the number of squares in the bottom row is $1 + 2(n - 1) = 2n - 1$.

The perimeter length P can be calculated by viewing the shape from four different directions and counting the number of horizontal or vertical segments. There are n vertical segments on each side (one for each row), making $2n$ in all. There are $2n - 1$ horizontal segments viewed from above, and the same number viewed from below, making $4n - 2$ in all. Therefore $P = 6n - 2$.

The area A is the total number of squares in the shape. But we may rearrange the shape into a square of side n, by dividing it into two pieces, rotating one of them by half a turn, and then recombining the squares, as shown in the diagrams.



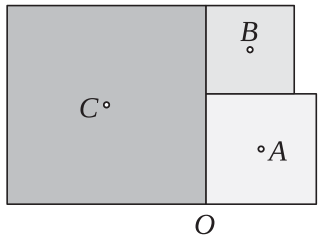


Thus $A = n^2$ and hence

$$\begin{aligned} 36A &= 36n^2 \\ &= (P + 2)^2 . \end{aligned}$$

M3. The diagram shows three squares with centres A, B and C. The point O is a vertex of two squares.

Prove that OB and AC are equal and perpendicular.

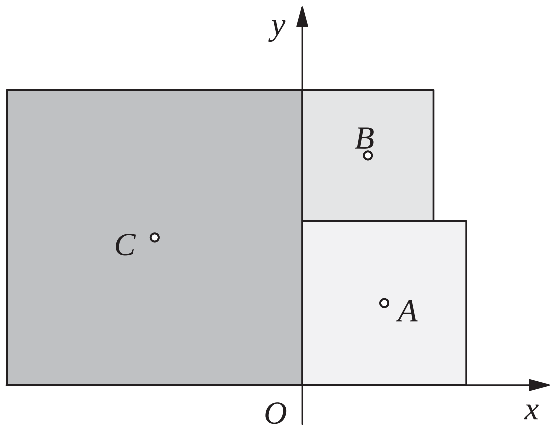


Solution

Let the three squares with centres A, B and C have sides of length $2a$,$2b$ and $2c$ respectively. Then

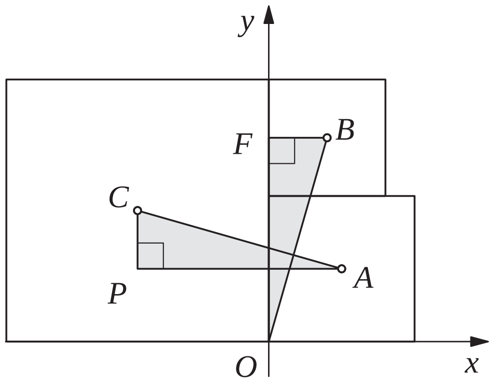
$$c = a + b. \tag{$*$}$$

Introduce coordinate axes as shown.



Then $A = (a, a)$, $B = (b,\ 2a + b)$ and $C = (-c,\ c)$.

Now consider the right-angled triangles OBF and ACP shown in the next diagram, where PA is parallel to the x-axis.



We have $BF = b$ and $OF = 2a + b$. Also $CP = c - a$ and $PA = c + a$. Using equation ($*$), we get $CP = b$ and $PA = 2a + b$. Thus $CP = BF$ and $PA = OF$. Using either congruent triangles (SAS) or Pythagoras' Theorem, we therefore obtain $CA = OB$.

Now the gradient of OB is $\dfrac{FO}{BF}$ and the gradient of AC is $-\dfrac{CP}{PA}$. It follows that the product of these gradients is -1, and hence OB and AC are perpendicular.

M4. What are the solutions of the simultaneous equations:

$$3x^2 + xy - 2y^2 = -5;$$
$$x^2 + 2xy + y^2 = 1\ ?$$

Solution

For convenience, we number the equations, as follows:

$$3x^2 + xy - 2y^2 = -5; \tag{1}$$
$$x^2 + 2xy + y^2 = 1. \tag{2}$$

By adding equation (1) to $5 \times$ equation (2), we obtain

$$8x^2 + 11xy + 3y^2 = 0,$$

that is,

$$(8x + 3y)(x + y) = 0.$$

Thus either $8x + 3y = 0$ or $x + y = 0$.

In the first case, from equation (2) we obtain $25x^2 = 9$, so that $x = \pm\frac{3}{5}$. Because $8x + 3y = 0$, if $x = \frac{3}{5}$, then $y = -\frac{8}{5}$, and if $x = -\frac{3}{5}$, then $y = \frac{8}{5}$.

Checking, we see that each of these is also a solution to equation (1).

In the second case, from equation (2) we get $0 = 1$, so this is impossible.

Hence the solutions of the given simultaneous equations are $x = \frac{3}{5}, y = -\frac{8}{5}$; and $x = -\frac{3}{5}, y = \frac{8}{5}$.

Alternative

We may factorise the left-hand side of each of the given equations as follows:

$$(3x - 2y)(x + y) = -5; \tag{3}$$
$$(x + y)^2 = 1. \tag{4}$$

From equation (4) it follows that $x + y = \pm 1$.

When $x + y = 1$, from equation (3) we get $3x - 2y = -5$. Solving these two linear simultaneous equations, we obtain $x = -\frac{3}{5}$ and $y = \frac{8}{5}$.

Similarly, when $x + y = -1$, we obtain $x = \frac{3}{5}$ and $y = -\frac{8}{5}$.

Checking, we see that each of these is also a solution to the original equations.

Hence the solutions of the given simultaneous equations are $x = \frac{3}{5}, y = -\frac{8}{5}$; and $x = -\frac{3}{5}, y = \frac{8}{5}$.

M5. The number of my hotel room is a three-digit integer. I thought that the same number could be obtained by multiplying together all of:

(i) one more than the first digit;

(ii) one more than the second digit;

(iii) the third digit.

Prove that I was mistaken.

Solution

Suppose that my hotel room number is 'abc', that is, $100a + 10b + c$.

If my belief is true, then

$$100a + 10b + c = (a + 1)(b + 1)c,$$

which we may rewrite as

$$\big(100 - (b + 1)c\big)a + (10 - c)b = 0. \qquad (*)$$

Now $(b + 1)c$ is at most 90 because b and c are digits. Also, a is at least 1 because the room number is a three-digit integer. Hence $\big(100 - (b + 1)c\big)a$ is at least 10; in particular, it is positive.

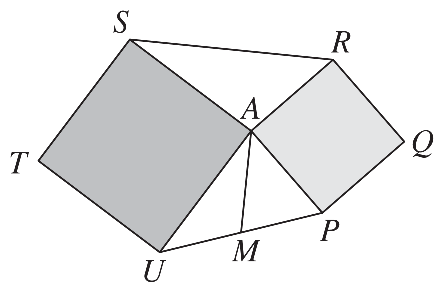
Furthermore, $10 - c$ is at least 1 and b is at least zero, so $(10 - c)b$ is at least zero.

It follows that the left-hand side of equation $(*)$ is positive, which is not possible.

Hence my belief is mistaken.

M6. The diagram shows two squares $APQR$ and $ASTU$, which have vertex A in common. The point M is the midpoint of PU.

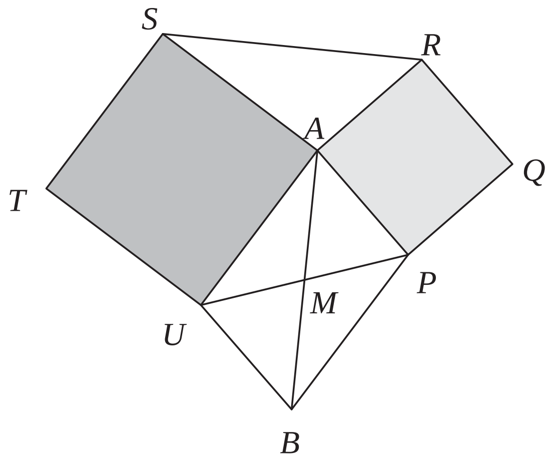
Prove that $AM = \frac{1}{2}RS$.



Solution

There are many methods, some of which use similar triangles or the cosine rule. The method we give below essentially only uses congruent triangles.

Let the point B be such that $AUBP$ is a parallelogram, as shown in the following diagram.



The diagonals of a parallelogram bisect one another. Therefore, because M is the midpoint of PU, it is also the midpoint of AB, in other words, $AM = \frac{1}{2}AB$.

Now we show that the triangles ABP and RSA are congruent.

Firstly, PA and AR are equal, because each is a side of the square $APQR$.

Also, $BP = UA$ because they are opposite sides of the parallelogram $AUBP$, and $UA = SA$ because each is a side of the square $ASTU$. Hence $BP = SA$.

Furthermore, because BP and UA are parallel, the angles BPA and PAU add up to 180° (allied angles, sometimes called interior angles). But $\angle RAP = 90°$ and $\angle UAS = 90°$ since each of them is an angle in a square. Then, by considering angles at the point A, we have

$$\angle SAR + 90° + \angle PAU + 90° = 360°,$$

so that the angles SAR and PAU add up to 180°. Hence $\angle BPA = \angle SAR$.

In the triangles ABP and RSA, we therefore have

$$\begin{aligned} PA &= AR, \\ \angle BPA &= \angle SAR \\ \text{and} \quad BP &= SA. \end{aligned}$$

Thus the triangles ABP and RSA are congruent (SAS). It follows that $AB = RS$.

But $AM = \frac{1}{2}AB$, so that $AM = \frac{1}{2}RS$, as required.