Solutions to the Olympiad Hamilton Paper

H1. No digit of the positive integer N is prime. However, all the single-digit primes divide N exactly.

What is the smallest such integer N?

Solution

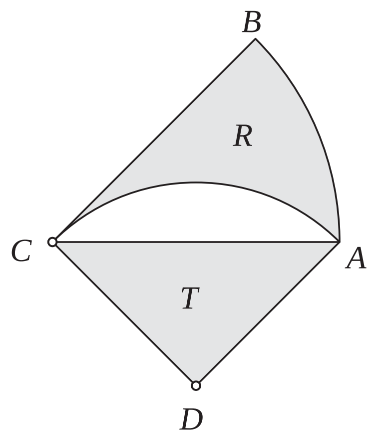
The single-digit primes are 2, 3, 5 and 7. Each of them divides N, so that $2 \times 3 \times 5 \times 7$ divides N. Written another way, this means that N is a multiple of $2 \times 3 \times 5 \times 7 = 210$.

But one of the digits of 210 is the prime 2, so N is not 210, and one of the digits of $2 \times 210 = 420$ is also 2, so N is not 420 either. Furthermore, one of the digits of $3 \times 210 = 630$ is the prime 3, so N is not 630. However, none of the digits of $4 \times 210 = 840$ is a prime, so N can be 840.

We have ruled out all the smaller options, therefore the smallest possible integer N is 840.

H2. The diagram shows two arcs. Arc AB is one eighth of a circle with centre C, and arc AC is one quarter of a circle with centre D. The points A and B are joined by straight lines to C, and A and C are joined by straight lines to D.

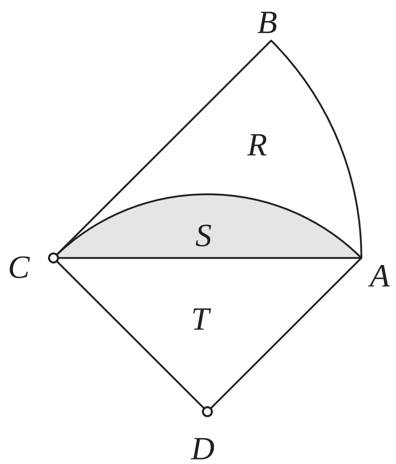
Prove that the area of the shaded triangle T is equal to the area of the shaded region R.



Solution

Let the radius DA be r, so that DC also equals r. Since arc AC is one quarter of a circle, angle $\angle CDA$ is 90°. Hence, using Pythagoras' Theorem in the triangle ACD, we obtain $CA^2 = 2r^2$.

Now consider the segment S bounded by the arc AC and the chord AC, shown shaded in the following diagram.



We shall combine this region with each of R and T—if the areas of the combined regions are equal, then the areas of R and T are equal.

The region obtained by combining R and S is one eighth of a circle with centre C and radius CA. Thus its area is

$$\begin{aligned} \tfrac{1}{8} \times \pi \times CA^2 &= \tfrac{1}{8} \times \pi \times 2r^2 \\ &= \tfrac{1}{4}\pi r^2. \end{aligned}$$

The region obtained by combining S and T is one quarter of a circle with centre D and radius DA. Thus its area is $\tfrac{1}{4}\pi r^2$.

Hence the areas of the regions obtained by combining S with each of T and R are equal. Therefore the area of T is equal to the area of R.

H3. Alex is given £1 by his grandfather and decides:

(i) to spend at least one third of the £1 on toffees at 5p each;

(ii) to spend at least one quarter of the £1 on packs of bubblegum at 3p each; and

(iii) to spend at least one tenth of the £1 on jellybeans at 2p each.

He only decides how to spend the rest of the money when he gets to the shop, but he spends all of the £1 on toffees, packs of bubblegum and jellybeans.

What are the possibilities for the number of jellybeans that he buys?

Solution

It follows from decision (i) that Alex spends at least 35p on toffees; it follows from decision (ii) that he spends at least 27p on bubblegum; and it follows from decision (iii) that he spends at least 10p on jellybeans. Therefore, out of the total £1 that he will spend, he has to decide how to spend 28p.

He may spend the whole 28p on jellybeans, which is an extra 14 jellybeans.

He cannot spend 26p or 24p on jellybeans, because he cannot spend the remaining money (2p or 4p) on the other items.

But he may spend any even amount from 22p downwards on jellybeans, since the remaining money would then be an even amount from 6p to 28p, and he is able to spend this on toffees or bubblegum (or both), as the following table shows.

Remaining money	Toffees at 5p	Bubblegum at 3p
6p	0	2
8p	1	1
10p	2	0
12p	0	4
14p	1	3
16p	2	2
18p	0	6
20p	1	5
22p	2	4
24p	0	8
26p	1	7
28p	2	6

Note that in some cases there are other ways to spend the money.

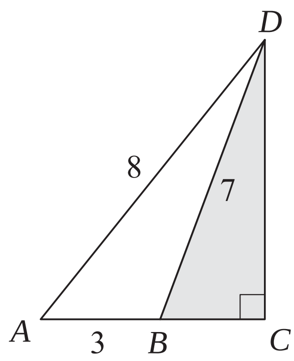
Thus the number of additional jellybeans that he may buy is a number from 0 to 11, or is 14.

But these are in addition to the five he buys as a result of decision (iii). Therefore the number of jellybeans that he buys is a number from 5 to 16, or is 19.

H4. The diagram shows a right-angled triangle *ACD* with a point *B* on the side *AC*.

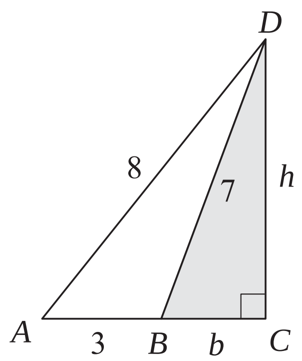
The sides of triangle *ABD* have lengths 3, 7 and 8, as shown.

What is the area of triangle *BCD*?



Solution

Let *BC* equal *b* and *CD* equal *h*, as shown in the following diagram.



Using Pythagoras' Theorem in both the triangle *BCD* and the triangle *ACD*, we get the two equations

$$b^2 + h^2 = 7^2 \tag{1}$$

and $$(b+3)^2 + h^2 = 8^2. \tag{2}$$

Subtracting equation (1) from equation (2), we obtain

$$(b+3)^2 - b^2 = 8^2 - 7^2.$$

Factorising the difference of two squares, we get

$$(b + 3 - b)(b + 3 + b) = (8 - 7)(8 + 7)$$

so that

$$3(2b + 3) = 15.$$

Therefore

$$b = 1.$$

Using equation (1), we now obtain $1 + h^2 = 7^2$, and so $h = \sqrt{48} = 4\sqrt{3}$.

Therefore the area of the triangle *BCD* is $\frac{1}{2} \times 1 \times 4\sqrt{3}$, that is, $2\sqrt{3}$.

H5. James chooses five different positive integers, each at most eight, so that their mean is equal to their median.

In how many different ways can he do this?

Solution

Since there are five different positive integers, the middle one is at least 3. But the five numbers are at most 8, so the middle one is at most 6.

However, the middle value is the median of the numbers, which we are told is equal to the mean. Let the integers, in increasing order, be a, b, c, d and e. Thus c is the median and therefore the mean.

Because c is the mean, $a + b + c + d + e = 5c$, and thus $(a + b) + (d + e) = 4c$. We know that $d + e$ is at most $7 + 8 = 15$, so that $a + b$ is at least $4c - 15$; also, $a + b$ is at most $(c - 2) + (c - 1) = 2c - 3$.

For each value of c, we first list the possible values of $a + b$. From $a + b$ we find $d + e$.

Finally, we list the possible values of a, b, d and e, using $0 < a < b < c < d < e$. The following table shows the results.

c	$a + b$	$d + e$	a	b	d	e
3	3	9	1	2	4	5
4	3	13	1	2	5	8
			1	2	6	7
	4	12	1	3	5	7
	5	11	2	3	5	6
5	5	15	1	4	7	8
			2	3	7	8
	6	14	2	4	6	8
	7	13	3	4	6	7
6	9	15	4	5	7	8

So altogether there are 10 ways for James to choose the integers.

H6. Tony multiplies together at least two consecutive positive integers. He obtains the six-digit number N. The left-hand digits of N are ‘47’, and the right-hand digits of N are ‘74’.

What integers does Tony multiply together?

Solution

An integer is divisible by 4 when the number formed from the rightmost two digits is a multiple of 4, and not otherwise. But 74 is not a multiple of 4, so N is not divisible by 4.

However, when two even numbers are multiplied together, the result is a multiple of 4. We conclude that Tony's list of consecutive integers does not include two even numbers.

There are two possibilities: either he has multiplied three consecutive integers ‘odd’, ‘even’, ‘odd’; or he has multiplied two consecutive integers. But when two consecutive integers are multiplied together the last digit is never 4—the only options for the last digits are $0 \times 1, 1 \times 2. 2 \times 3, 3 \times 4, 4 \times 5, 5 \times 6, 6 \times 7, 7 \times 8, 8 \times 9, 9 \times 0$, so the result ends in 0, 2, 6, 2, 0, 0, 2, 6, 2 or 0.

We are therefore trying to find an odd integer n such that $n \times (n + 1) \times (n + 2) =$ ‘47… 74’. We also know that $n + 1$ is not a multiple of 4.

Now $81 \times 82 \times 83$ fails because it ends in 6. It is also too big, since it is bigger than $80 \times 80 \times 80 = 512\,000$.

Also $73 \times 74 \times 75$ fails because it ends in 0. It is also too small, since it is smaller than $75 \times 75 \times 75 = 421\,875$.

The only remaining possibility is $77 \times 78 \times 79 = 474\,474$, which works.