Solutions to the Olympiad Cayley Paper

C1. How many three-digit multiples of 9 consist only of odd digits?

Solution

We use the fact that 'an integer is a multiple of 9 when the sum of its digits is a multiple of 9, and not otherwise'.

Consider a three-digit integer with the required properties. Each digit is between 0 and 9, and none of them is zero, so the sum of the digits is between 1 and 27. Since we want the integer to be a multiple of 9, the sum of the digits is therefore 9, 18 or 27.

However, it is not possible to write 18 as a sum of three odd numbers, and the only way of making the sum of the digits equal to 27 is with 999, which is thus one possible integer. But the remaining question is 'how can we make the sum of the digits equal to 9?'

If one of the digits is 1, then we can make the remaining 8 in two ways:

1 + 7

giving the three integers 117, 171 and 711;

3 + 5

giving the six integers 135, 153, 315, 351, 513 and 531.

If we do not use a 1, then the only possible integer is 333.

Hence there are eleven three-digit multiples of 9 consisting only of odd digits.

C2. In a 6×6 grid of numbers:

(i) all the numbers in the top row and the leftmost column are the same;

(ii) each other number is the sum of the number above it and the number to the left of it;

(iii) the number in the bottom right corner is 2016.

What are the possible numbers in the top left corner?

Solution

Suppose we start by letting the number in the top left corner be t. Since, by rule (i), the numbers in the top row and left-hand column are all the same, we can fill them all in as t. Then we can use rule (ii) to fill in the rest of the grid, as shown.

t	t	t	t	t	t
t	$2t$	$3t$	$4t$	$5t$	$6t$
t	$3t$	$6t$	$10t$	$15t$	$21t$
t	$4t$	$10t$	$20t$	$35t$	$56t$
t	$5t$	$15t$	$35t$	$70t$	$126t$
t	$6t$	$21t$	$56t$	$126t$	$252t$

From rule (iii), we now obtain

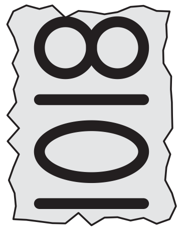
$$252t = 2016.$$

Dividing each side by 252, we get

$$t = 8.$$

So the only possible number in the top left corner is 8.

C3. All the telephone numbers in Georgetown have six digits and each of them begins with the digits 81. Kate finds the scrap of paper shown, with part of Jenny's telephone number on it.



How many different possibilities are there for Jenny's telephone number?

Solution

The scrap of paper could read ‘1018’ or ‘8101’.

Let us start by working out which parts of the phone number Kate could have been seeing. We show the scrap of paper as a rectangle.

8$\boxed{\textbf{1018}}$ ***d***

There are ten possibilities for the unknown digit *d*.

81$\boxed{\textbf{1018}}$

There is only one possibility.

$\boxed{\textbf{8101}}$ ***cd***

There are one hundred possibilities for the pair of unknown digits *c* and *d*.

81$\boxed{\textbf{8101}}$

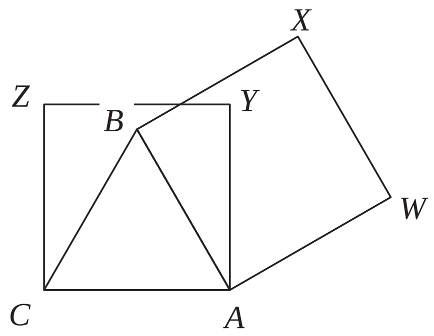
There is only one possibility.

However, all the possibilities in the first case—‘81018*d*’—are also counted under the third case—‘8101*cd*’.

Hence there are in fact 102 possibilities for Jenny's telephone number.

C4. The diagram shows an equilateral triangle ABC and two squares $AWXB$ and $AYZC$.

Prove that triangle AXZ is equilateral.



Solution

We start by studying the angles within the square $AYZC$, each of whose angles is 90°. Since the triangle ABC is equilateral, each of its angles is 60°. Hence

$$\begin{aligned} \angle BAY &= \angle CAY - \angle CAB \\ &= 90° - 60° \\ &= 30°. \end{aligned}$$

Also, $\angle ZAY = 45°$ since ZA is a diagonal of the square.

Now let us move over to the other square $AWXB$. We have $\angle BAX = 45°$ since AX is a diagonal.

Therefore

$$\begin{aligned} \angle ZAX &= \angle ZAY - \angle BAY + \angle BAX \\ &= 45° - 30° + 45° \\ &= 60°. \end{aligned}$$

Also, the two squares have the same size, because the triangle ABC is equilateral and so $AB = AC$. Thus AZ and AX are diagonals of squares of the same size, and hence $AZ = AX$.

It follows that the triangle AXZ is an equilateral triangle—it has two equal sides and the angle between them is 60°.

C5. Dean wishes to place the positive integers $1, 2, 3, \ldots, 9$ in the cells of a 3×3 square grid so that:

(i) there is exactly one number in each cell;

(ii) the product of the numbers in each row is a multiple of four;

(iii) the product of the numbers in each column is a multiple of four.

Is Dean's task possible? Prove that your answer is correct.

Solution

We claim that Dean's task is impossible, and will prove our claim by contradiction.

Suppose Dean has placed the integers according to the given rules.

Now only one row contains 4, and only one row contains 8. But there are three rows, so there is at least one row R which contains neither 4 nor 8. From rule (ii), the product of the numbers in R is a multiple of 4 and hence R contains both 2 and 6.

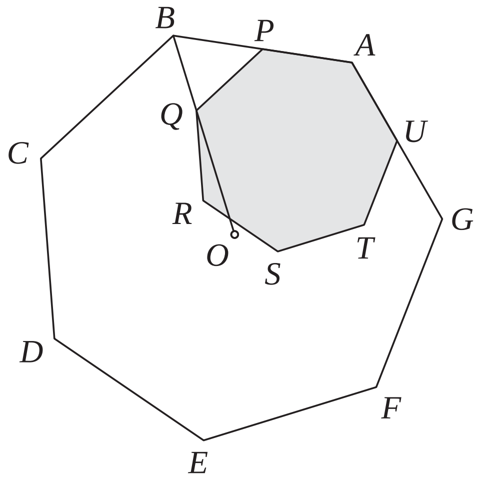
In a similar way, there is at least one column C that contains neither 4 nor 8. From rule (iii), the product of the numbers in C is a multiple of 4 and hence C contains both 2 and 6.

As a result, both 2 and 6 are in the same row, and in the same column, which contradicts rule (i).

Therefore Dean's task is not possible.

C6. The diagram shows two regular heptagons *ABCDEFG* and *APQRSTU*. The vertex *P* lies on the side *AB* (and hence *U* lies on the side *GA*). Also, *Q* lies on *OB*, where *O* is the centre of the larger heptagon.

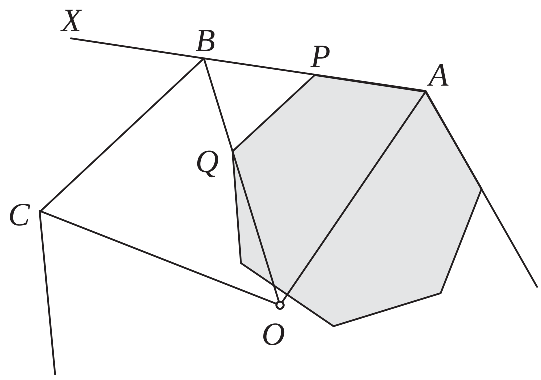
Prove that $AB = 2AP$.



Solution

We will show that $PQ = PB$. Then it will follow that $AP = PB$, because *AP* and *PQ* are two sides of a regular heptagon, and hence that $AB = 2AP$.

Draw the lines *OA* and *OC* and extend *AB* to *X*, as shown.



Now the angles *CBX* and *QPB* are equal because each is an exterior angle of a regular heptagon. Thus *BC* is parallel to *PQ* (corresponding angles converse), and therefore angle *OBC* is equal to angle *BQP* (alternate angles).

Consider triangles *OAB* and *OCB*. The side *OB* is common, the sides *OA* and *OC* are equal because *O* is the centre of the larger heptagon, and the sides *AB* and *BC* are equal because the heptagon is regular.

Thus triangles *OAB* and *OCB* are congruent (SSS), and therefore angle *OBC* is equal to angle *OBA*, which is the same as angle *PBQ*.

Hence the angles *PBQ* and *BQP* are equal. Therefore, using 'sides opposite equal angles are equal' in the triangle *BQP*, we get $PQ = PB$ as required.