Intermediate Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

1 February 2018

These solutions augment the printed solutions that we send to schools. For convenience, the solutions sent to schools are confined to two sides of A4 paper and therefore in many cases are rather short. The solutions given here have been extended. In some cases we give alternative solutions, and we have included some exercises for further investigation. We welcome comments on these solutions. Please send them to enquiry@ukmt.org.uk.

The Intermediate Mathematical Challenge (IMC) is a multiple-choice paper. For each question, you are presented with five options, of which just one is correct. It follows that often you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the IMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the question without any alternative answers. So for each question we have included a complete solution which does not use the fact that one of the given alternatives is correct. Thus we have aimed to give full solutions with all steps explained (or, occasionally, left as an exercise). We therefore hope that these solutions can be used as a model for the type of written solution that is expected when a complete solution to a mathematical problem is required (for example, in the Intermediate Mathematical Olympiad and similar competitions).

Enquiries about the Intermediate Mathematical Challenge should be sent to:

IMC, UKMT, School of Mathematics Satellite, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 enquiry@ukmt.org.uk www.ukmt.org.uk

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B	B	C	D	B	A	B	A	E	E	B	E	C	C	E	C	D	A	D	E	D	E	A	B	D

1. Which of these is the sum of the cubes of two consecutive integers?				
A 4	B 9	C 16	D 25	E 36

Solution **B**

We see that

$$9 = 1^3 + 2^3,$$

and hence 9 is the sum of the cubes of two consecutive integers.

Note

In this context, it is sufficient to find one option that is correct, as we are entitled to assume that in the IMC there is just one correct option.

It is, in fact, straightforward to check by listing the first few cubes that none of the other options is correct. We leave this as an exercise.

For investigation

1.1 Show that none of the other numbers given as options, that is, 4, 16, 25 and 36, is the sum of two consecutive cubes.

1.2 Calculate the following sums.

(a) $1^3 + 2^3 + 3^3$.

(b) $1^3 + 2^3 + 3^3 + 4^3$.

(c) $1^3 + 2^3 + 3^3 + 4^3 + 5^3$.

1.3 What do you notice about your answers to Problem 1.2?

1.4 Find a general formula for the sum of the cubes of the first n positive integers.

1.5 Prove that the formula given by your answer to Problem 1.4 is correct. Deduce that the sum of the first n positive cubes is always a square.

1.6 Other than $1^3 + 2^3 + 3^3 = 6^2$, there is just one example where the sum of the cubes of three consecutive positive integers is a square. Can you find this example?

Note

It is known that $1^3 + 2^3 = 3^2$ is the only case where the sum of the cubes of two consecutive positive integers is a square. However, we do not know a short elementary proof of this fact. If you know of one, please send it to us (at enquiry@ukmt.org.uk).

In order to establish this fact we would need to show that the only solution of the equation $x^3 + (x + 1)^3 = y^2$ in which x and y are positive integers is $x = 1$, $y = 3$.

The equation $x^3 + (x + 1)^3 = y^2$ is an example of an equation of an *elliptic curve*. The theory of the integer solutions of such equations is complicated. The theory of elliptic curves was one of the ingredients of the proof by Andrew Wiles of "Fermat's Last Theorem". By the way, elliptic curves are *not* ellipses. Search the web if you wish to know how they come to have this name.

1.7 Sketch the graph of the curve given by the equation $x^3 + (x + 1)^3 = y^2$.

2. How many of these four integers are prime?

1	11	111	1111

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

We begin by recalling the definition of a prime:

A positive integer n is said to be *prime* if $n \neq 1$ and the only divisors of n are 1 and n.

It follows from the definition that 1 is not prime. It is easy to see that 11 is prime. Since $111 = 3 \times 37$, it follows that 111 is not prime. Since $1111 = 11 \times 101$, it follows that 1111 is not prime.

Therefore just one of the given integers is prime.

NOTE

It is just a convention not to regard 1 as a prime number, but it is a standard convention that it is important to remember.

"In case you're wondering, I'll mention that 1 is not prime. That's not because of some interesting philosophical point, or because it's only got one number that divides into it rather than two, or anything like that. Definitions (such as that of a prime number) don't get handed to mathematicians on stone tablets. Rather, part of the job of mathematicians is to make good definitions, ones that lead to interesting mathematics, and it turns out to be better to define 1 not to be prime." Vicky Neale, *Closing the Gap: the quest to understand prime numbers*, Oxford University Press, 2017.

FOR INVESTIGATION

2.1 If n can be factorized as $r \times s$, where r and s are integers, at least one of r and s will be at most $\sqrt{n}$. [Why?] Therefore, to test whether an integer n, with $n > 1$, is prime it is sufficient to see whether n is divisible by a prime p in the range $2 \leq p \leq \sqrt{n}$. The integer n is prime if, and only if, it has no prime divisor in this range.

Use this criterion to show that 11 is prime and to test whether the following are primes.
(a) 91, (b) 107, (c) 899, (d) 901, (e) 907.

2.2 Find the prime factorizations of the following integers.
(a) 11 111, (b) 111 111, (c) 1 111 111.

2.3 A positive integer which may be written as a string of 1s is called a *repunit* (short for *rep*eated *unit*s). Show that if n is not a prime number, then the repunit consisting of a string of n 1s is not a prime number.

2.4 We have already noted that the repunit 111 consisting of a string of three 1s is not a prime. This shows that the converse of Problem 2.3 is not true in general; if n is a prime, the repunit consisting of a string of n 1s need not be a prime.

Find the least prime number n, with $n > 2$, such that the repunit consisting of a string of n 1s is a prime. (Note: as the repunits you will need to check are quite large, you will need an electronic aid to test whether they are primes.)

3. In September 2016 a polymer £5 note was introduced. The Bank of England issued 440 million of them.

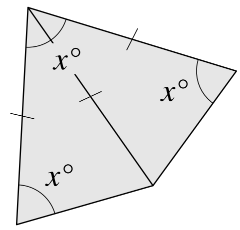
What is the total face value of all these notes?

A £220 000 000 B £440 000 000 C £2 200 000 000
D £4 400 000 000 E £22 000 000 000

SOLUTION **C**

One million is 1 000 000. Therefore 440 million is 440 000 000. It follows that the total face value of 440 million £5 notes is £$(5 \times 440\,000\,000)$ = £2 200 000 000.

4. A kite is made by joining two congruent isosceles triangles, as shown.

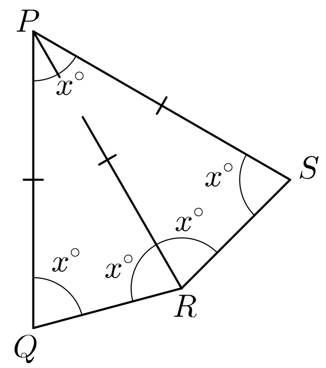


What is the value of x?

A 36 B 54 C 60 D 72 E 80

SOLUTION **D**

Let the vertices of the kite be labelled as shown.



The question tells us that the triangles PQR and PRS are isosceles. It follows that $\angle QRP = \angle RQP = x°$ and that $\angle SRP = \angle RSP = x°$. Hence $\angle SRQ = 2x°$.

The sum of the angles of a quadrilateral is 360°. [You are asked to prove this in Problem 4.1 below.] Therefore, from the quadrilateral $PQRS$ we have

$$x° + x° + 2x° + x° = 360°.$$

Therefore $5x° = 360°$ and hence $x = 72$.

FOR INVESTIGATION

4.1 Prove that the sum of the angles of a quadrilateral is 360°.

4.2 The solution also uses the following theorem

The angles at the base of an isosceles triangle are equal to one another.

Find a proof of this theorem.

4.3 The theorem stated in Problem 4.2 is Proposition 5 of Euclid's *Elements*, Book 1. In the days when Euclid's *Elements* was used as a textbook in schools this proposition was known as the *pons asinorum*. Find out what this means, and why the theorem was given this name.

5. The adult human body has 206 bones. Each foot has 26 bones.

Approximately what fraction of the number of bones in the human body is found in one foot?

A $\frac{1}{6}$ B $\frac{1}{8}$ C $\frac{1}{10}$ D $\frac{1}{12}$ E $\frac{1}{20}$

SOLUTION **B**

The required fraction is

$$\frac{26}{206} = \frac{13}{103}.$$

Since $103 = 7 \times 13 + 12$, we have

$$\frac{13}{103} = \frac{13}{7 \times 13 + 12} = \frac{1}{7 + \frac{12}{13}}.$$

Because the integer closest to $7 + \frac{12}{13}$ is 8, we deduce that $\frac{26}{206}$ is approximately equal to $\frac{1}{8}$.

FOR INVESTIGATION

5.1 For which integer n does the fraction $\frac{1}{n}$ give the best approximation to $\frac{23}{2018}$?

6. In 2014, in Boston, Massachusetts, Eli Bishop set a world record for the greatest number of claps per minute. He achieved 1020 claps in one minute.

How many claps is that per second?

A 17 B 16.5 C 16 D 15.5 E 15

SOLUTION **A**

There are 60 seconds in one minute. Therefore 1020 claps per minute is the same as $1020 \div 60$ claps per second.

Now

$$\frac{1020}{60} = \frac{102}{6} = 17.$$

Therefore Eli Bishop clapped at 17 claps per second.

FOR INVESTIGATION

6.1 Suppose that each time Eli clapped he moved each of his hands through 10 cm, that is, 5 cm in moving the hands apart and then another 5 cm bringing them together again. On this assumption, what was the average speed of movement of his hands in kilometres per hour?

7. How many two-digit squares have the property that the product of their digits is also a square?

A 0 B 1 C 2 D 3 E 4

SOLUTION **B**

In the following table we have listed the two-digit squares together with the products of their digits.

n	n^2	product of the digits of n^2
4	16	6
5	25	10
6	36	18
7	49	36
8	64	24
9	81	8

From this table we see that the only two-digit square which has the property that the product of its digits is a square is 49. Therefore the number of two-digit squares with this property is 1.

FOR INVESTIGATION

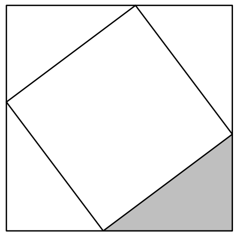
7.1 How many three-digit squares have the property that the product of their digits is the square of a positive integer?

7.2 There are just two four-digit squares which have the property that the product of their digits is the square of a positive integer. Can you find them?

8. The diagram shows a square of perimeter 20 cm inscribed inside a square of perimeter 28 cm.

What is the area of the shaded triangle?

A $6\,\text{cm}^2$ B $7\,\text{cm}^2$ C $8\,\text{cm}^2$ D $9\,\text{cm}^2$ E $10\,\text{cm}^2$



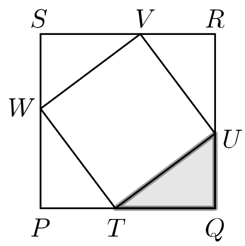
SOLUTION **A**

A quick solution in the context of the IMC goes as follows:
Clearly the four triangles in the corners of the larger square are congruent, and so have the same area. The squares have perimeters of lengths 20 cm and 28 cm. Hence their side lengths are 5 cm and 7 cm, so their areas are $5^2\,\text{cm}^2 = 25\,\text{cm}^2$ and $7^2\,\text{cm}^2 = 49\,\text{cm}^2$. The area of the shaded region is one quarter of the difference between the areas of the squares. So the area of the shaded region is $\frac{1}{4}(49 - 25)\,\text{cm}^2 = 6\,\text{cm}^2$.

However, for a full solution we need to spell out in detail how we know that the four triangles are congruent. We give two methods for doing this.

METHOD 1

We let P, Q, R and S be the vertices of the larger square, and T, U, V and W be the vertices of the smaller square, arranged as shown in the figure.



Because $PQRS$ is a square, $\angle TQU = 90°$. Therefore, applying the fact that the sum of the angles in a triangle is 180° to triangle TQU, we have $\angle QTU + \angle TUQ + 90° = 180°$. Hence,

$$\angle TUQ = 90° - \angle QTU. \quad (1)$$

Because $TUVW$ is a square, $\angle UTW = 90°$. Therefore, applying the fact that the sum of the angles on a line is 180° to the angles at T, we have $\angle QTU + 90° + \angle PTW = 180°$. Hence

$$\angle PTW = 90° - \angle QTU. \quad (2)$$

By (1) and (2), $\angle TUQ = \angle PTW$. Similarly, $\angle QTU = \angle TWP$. Because $TUVW$ is a square, $TU = WT$. It follows that the triangles TQU and WPT are congruent. Similarly, these triangles are also congruent to the triangles VSW and URV. Therefore all these four triangles have the same area. It follows that the area of the shaded region is one quarter of the area of the square $PQRS$ less the area of the square $TUVW$.

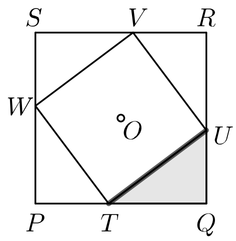
The square $PQRS$ has perimeter 28 cm and hence it has side length 7 cm. The square $TUVW$ has perimeter 20 cm and hence side length 5 cm. Therefore these squares have areas $49\,\text{cm}^2$ and $25\,\text{cm}^2$, respectively.

Hence the area of the shaded region is given by

$$\tfrac{1}{4}(49\,\text{cm}^2 - 25\,\text{cm}^2) = \tfrac{1}{4}(24\,\text{cm}^2) = 6\,\text{cm}^2.$$

METHOD 2

Let O be the centre of the square $TUVW$. It is left as an exercise (see Problem 8.1) to show that O is also the centre of the square $PQRS$.



Consider the rotation of both squares through a quarter turn anticlockwise about the point O.

The effect of this rotation is to move P to Q, Q to R, R to S and S to P. It also moves T to U, U to V, V to W and W to T.

It follows that this rotation moves the triangle PTW to the position of the triangle QUT, QUT to RVU, RVU to SWV and SWV to PTW. It follows that these four triangles all have the same area.

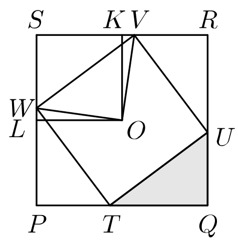
It then follows, as in Method 1, that the area of each of these triangles is a quarter of the difference of the areas of the two squares. Hence the area of the shaded triangle is $6\,\text{cm}^2$.

FOR INVESTIGATION

8.1 Show that the two squares $PQRS$ and $TUVW$ have the same centre.

NOTE

One way to do this is to suppose that O is the centre of the square $TUVW$. We then let K be the point where the perpendicular from O to SR meets SR, and let L be the point where the perpendicular from O to PS meets PS.



Now show that the triangles OVK and OLW are congruent. It will follow that $OK = OL$.

Deduce from this that O lies on the diagonal SQ.

In a similar way it may be shown that O lies on the diagonal PR, and hence that O is the centre of the square $PQRS$.

9. Which integer n satisfies $\frac{3}{10} < \frac{n}{20} < \frac{2}{5}$?

A 3 B 4 C 5 D 6 E 7

SOLUTION **E**

If we multiply both sides of an inequality by a *positive* number, we obtain an equivalent inequality.

Therefore, by multiplying by the positive number 20, we see that the given inequalities are equivalent to

$$6 < n < 8.$$

The only integer that satisfies these inequalities is 7.

10. Which of these integers cannot be expressed as the difference of two squares?

A 5 B 7 C 8 D 9 E 10

SOLUTION **E**

METHOD 1

We see that $5 = 3^2 - 2^2$, $7 = 4^2 - 3^2$, $8 = 3^2 - 1^2$ and $9 = 5^2 - 4^2$.

Therefore the first four integers given as options can be expressed as the difference of two squares. In the context of the IMC it is safe to deduce that the remaining option, 10, cannot be expressed as the difference between two squares.

METHOD 2

The first six squares are 1, 4, 9, 16, 25, 36. Once we get beyond 25 the difference between consecutive squares is greater than 10. Hence if 10 could be expressed as the difference of squares, the two squares in question would have both to be at most 25. However, it is straightforward to check that there does not exist a pair of numbers taken from the list 1, 4, 9, 16, 25 whose difference is 10.

Therefore we deduce that 10 cannot be expressed as the difference of two squares.

A third approach to Question 10 is to find exactly which positive integers are the difference of two squares. Problems 10.1 and 10.2 lead you in this direction.

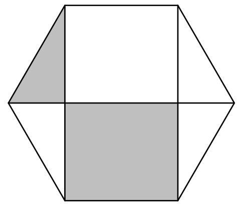
FOR INVESTIGATION

10.1 Prove that every odd integer can be expressed as the difference of two squares.

10.2 Determine which even integers can be expressed as the difference of two squares.

10.3 Deduce the answer to Question 10 from your answers to the above two problems.

11. The diagram shows a regular hexagon which has been divided into six regions by three of its diagonals. Two of these regions have been shaded. The total shaded area is $20\,\text{cm}^2$.

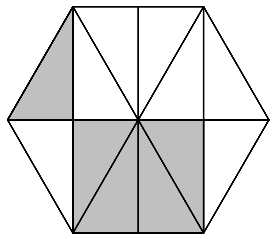


What is the area of the hexagon?

A $40\,\text{cm}^2$ B $48\,\text{cm}^2$ C $52\,\text{cm}^2$ D $54\,\text{cm}^2$ E $60\,\text{cm}^2$

Solution **B**

Let the area of the hexagon be $a\,\text{cm}^2$.



In the figure on the right we have added two more of the diagonals of the hexagon, and a line joining the midpoints of two opposite edges.

The lines in the figure divide the hexagon into 12 congruent triangles. The shaded region is covered by 5 of these triangles. Hence the shaded area makes up five-twelfths of the total area of the hexagon.

The area of the shaded region is $20\,\text{cm}^2$. Therefore,

$$\frac{5}{12}a = 20.$$

It follows that

$$a = \frac{12}{5} \times 20 = 48.$$

Hence, the area of the hexagon is $48\,\text{cm}^2$.

For investigation

11.1 Prove the assertion made in the above solution that the 12 triangles in the above figure are all congruent.

11.2 What is the length of each of the edges of the hexagon?

12. Someone has switched the numbers around on Harry's calculator!

The numbers should be in the positions shown in the left-hand diagram, but have been switched to the positions in the right-hand diagram.

7 8 9	9 8 7
4 5 6	6 5 4
1 2 3	3 2 1

Which of the following calculations will *not* give the correct answer when Harry uses his calculator?

A 79×97 B 78×98 C 147×369 D 123×321 E 159×951

Solution **E**

Six of the numbers have been switched round in pairs, as follows:
$1 \longleftrightarrow 3$ $4 \longleftrightarrow 6$ $7 \longleftrightarrow 9$. The numbers 2, 5 and 8 are unchanged.
We consider the effect of these interchanges on each of the given calculations in turn.

79×97 becomes 97×79. We have $97 \times 79 = 79 \times 97$.

78×98 becomes 98×78. We have $98 \times 78 = 78 \times 98$.

147×369 becomes 369×147. We have $369 \times 147 = 147 \times 369$.

123×321 becomes 321×123 We have $321 \times 123 = 123 \times 321$.

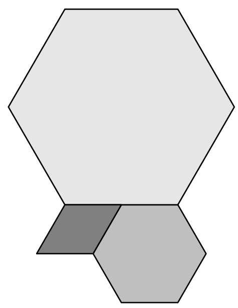
159×951 becomes 357×753. However $753 \times 357 \neq 159 \times 951$.

Therefore it is calculation E that does not give the correct answer.

13. The diagram shows a rhombus and two sizes of regular hexagon.

What is the ratio of the area of the smaller hexagon to the area of the larger hexagon?

A 1 : 2 B 1 : 3 C 1 : 4 D 1 : 8 E 1 : 9



Solution **C**

From the figure in the question we see that the rhombus and the smaller hexagon have the same side lengths. We also see that the side length of the larger hexagon is equal to the sum of the side lengths of the rhombus and the smaller hexagon. Therefore the ratio of the side length of the smaller hexagon to that of the larger hexagon is 1 : 2.

Hence the ratio of the area of the smaller hexagon to that of the larger hexagon is $1^2 : 2^2$, that is, 1 : 4.

For investigation

13.1 Explain why it is true that if the lengths of corresponding edges of two similar polygons are in the ratio $a : b$, then their areas are in the ratio $a^2 : b^2$.

14. Which of these is equal to $\frac{10}{9} + \frac{9}{10}$?

A 1 B 2 C $2.0\dot{1}$ D $2.\dot{1}$ E $2.\dot{2}$

SOLUTION **C**

Putting the two fractions over a common denominator, we have

$$\frac{10}{9} + \frac{9}{10} = \frac{10 \times 10 + 9 \times 9}{90} = \frac{100 + 81}{90} = \frac{181}{90} = 2 + \frac{1}{90}.$$

Now, as a decimal, $\frac{1}{9}$ is the recurring decimal $0.\dot{1}$. Hence, as a decimal, $\frac{1}{90}$ is $0.0\dot{1}$. Therefore,

$$2 + \frac{1}{90} = 2 + 0.0\dot{1} = 2.0\dot{1}.$$

FOR INVESTIGATION

14.1 Work out each of the fractions $\frac{10}{9}$ and $\frac{9}{10}$ as decimals. Then add the two decimals.

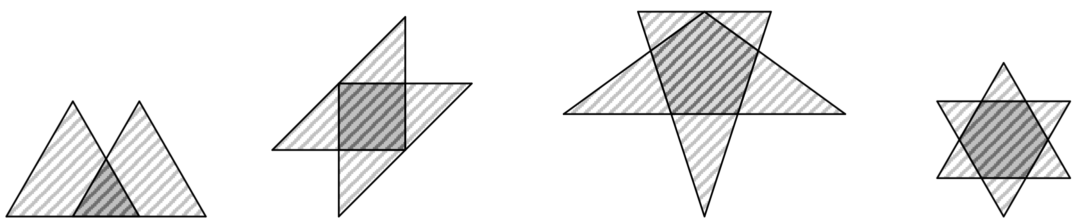
15. How many of these four shapes could be the shape of the region where two triangles overlap?

equilateral triangle	square	regular pentagon	regular hexagon

A 0 B 1 C 2 D 3 E 4

SOLUTION **E**

We see from the figures below that each of the four given shapes can be obtained as the intersection of two triangles.



equilateral triangle square regular pentagon regular hexagon

FOR INVESTIGATION

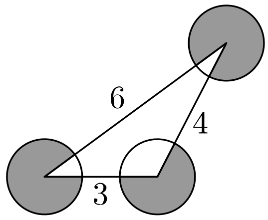
15.1 Find examples of shapes that *cannot* be the shape of the region where two triangles overlap.

16. The diagram shows a triangle with edges of length 3, 4 and 6.

A circle of radius 1 is drawn at each vertex of the triangle.

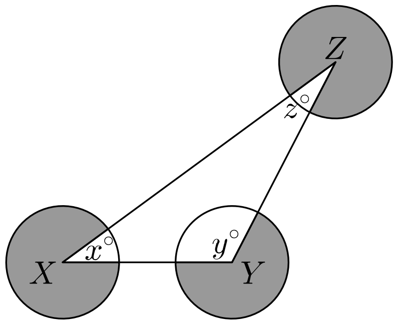
What is the total shaded area?

A 2π B $\frac{9\pi}{4}$ C $\frac{5\pi}{2}$ D $\frac{11\pi}{4}$ E 3π



SOLUTION **C**

We let the vertices of the triangle be X, Y, Z and the angles be $x°$, $y°$, $z°$, as shown in the figure.



Because the circle with centre X has radius 1, it has area $\pi 1^2$, that is, π. Of this area, the fraction that is not shaded is $\frac{x}{360}$. Hence the fraction that is shaded is $1-\frac{x}{360}$. Hence the area that is shaded is $\left(1-\frac{x}{360}\right)\pi$. Similarly the areas of the other two circles that are shaded are $\left(1-\frac{y}{360}\right)\pi$ and $\left(1-\frac{z}{360}\right)\pi$.

It follows that the total area that is shaded is given by

$$\left(1-\frac{x}{360}\right)\pi+\left(1-\frac{y}{360}\right)\pi+\left(1-\frac{z}{360}\right)\pi=\left(3-\frac{x+y+z}{360}\right)\pi.$$

Now, because $x°$, $y°$ and $z°$ are the three angles of a triangle, $x+y+z=180$. Hence

$$\left(3-\frac{x+y+z}{360}\right)\pi=\left(3-\frac{180}{360}\right)\pi=\left(3-\frac{1}{2}\right)\pi=\frac{5\pi}{2}.$$

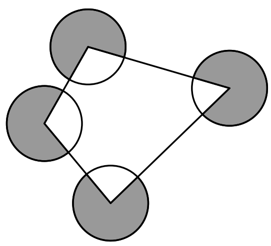
Hence the total shaded area is $\frac{5\pi}{2}$.

[Note that this answer is the same whatever the lengths of the edges of the triangle provided that they are sufficiently large that the circles do not overlap.]

FOR INVESTIGATION

16.1 The figure shows four circles each with radius 1 whose centres are the vertices of a quadrilateral.

What is the total shaded area?



16.2 What is the total shaded area in the general case where the quadrilateral is replaced by a polygon with n vertices?

17. How many three-digit numbers are increased by 99 when their digits are reversed?

A 4 B 8 C 10 D 80 E 90

SOLUTION **D**

We use the notation 'cba' to denote the number with hundreds digit c, tens digit b and units digit a. Thus 'cba' denotes the number $100c + 10b + a$.

The number obtained by reversing the digits of 'cba' is 'abc'. The increase that results when its digits are reversed is therefore

$$\text{`}abc\text{'} - \text{`}cba\text{'} = (100a + 10b + c) - (100c + 10b + a) = 99(a - c).$$

Hence, the condition that the three-digit number 'cba' is increased by 99 when its digits are reversed is that $a - c = 1$, that is, $a = c + 1$.

Now let 'cba' be one such three-digit number. The hundreds digit c cannot be 0. Because $a \leq 9$, and $a = c + 1$, we see that $c \leq 8$. Therefore there are 8 choices for the hundreds digit c, namely, 1, 2, 3, 4, 5, 6, 7 and 8. The tens digit b can be any of the 10 digits. However, once c has been chosen, there is just 1 choice for a, since $a = c + 1$.

Therefore the total number of choices for a three-digit number 'cba' which is increased by 99 when its digits are reversed is $8 \times 10 \times 1 = 80$.

FOR INVESTIGATION

17.1 How many three-digit numbers are increased by 100 when their digits are reversed?

17.2 How many three-digit numbers are increased by 198 when their digits are reversed?

17.3 How many four-digit numbers are increased by 1089 when their digits are reversed?

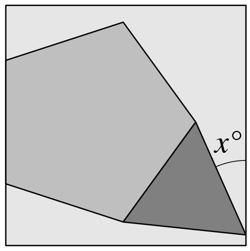
17.4 Let n be a three-digit number, and let n^* be the number obtained from n by reversing the order of its digits. What values can $n - n^*$ take?

17.5 Let n be a four-digit number, and let n^* be the number obtained from n by reversing the order of its digits. What values can $n - n^*$ take?

18. The diagram shows a regular pentagon and an equilateral triangle placed inside a square.

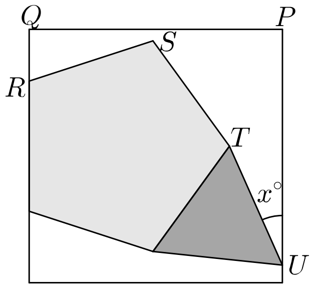
What is the value of x?

A 24 B 26 C 28 D 30 E 32



Solution **A**

We let P, Q, R, S, T and U be the vertices as labelled in the figure on the right.



Our strategy is to calculate all the angles in the hexagon $PQRSTU$ other than the angle x°. We can then use the fact that the sum of the angles in a hexagon is 720° to work out the value of x.

We let $\hat{P}$, $\hat{Q}$, $\hat{R}$, $\hat{S}$, $\hat{T}$ and $\hat{U}$ be the interior angles of the hexagon $PQRSTU$ at the corresponding vertices.

First, $\hat{P} = \hat{Q} = 90^\circ$ because they are angles of a square.

Because it is an exterior angle of a regular pentagon, $\hat{R} = 72^\circ$.

Because the interior angle of a regular pentagon is 108°, we have $\hat{S} = 360^\circ - 108^\circ = 252^\circ$.

Because the interior angle of a regular pentagon is 108° and the interior angle of an equilateral triangle is 60°, we have $\hat{T} = 360^\circ - (108^\circ + 60^\circ) = 192^\circ$.

Finally, $\hat{U} = x^\circ$.

Since the sum of the angles in a hexagon is 720°, it follows that

$$\hat{P} + \hat{Q} + \hat{R} + \hat{S} + \hat{T} + \hat{U} = 720^\circ.$$

Therefore, by the above facts about the angles in this equation,

$$90^\circ + 90^\circ + 72^\circ + 252^\circ + 192^\circ + x^\circ = 720^\circ,$$

and hence

$$696 + x = 720.$$

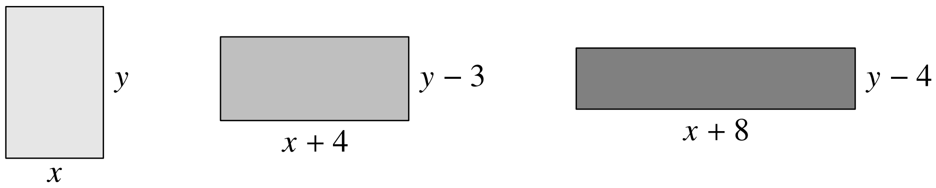
It follows that $x = 24$.

For investigation

18.1 Prove the fact used in the above solution that the exterior angle of a regular pentagon is 72°.

18.2 Prove the fact used in the above solution that the sum of the angles in a hexagon is 720°.

19. The three rectangles shown below all have the same area.



What is the value of $x + y$?

A 4 B 6 C 8 D 10 E 12

SOLUTION **D**

METHOD 1

Because the rectangle with side lengths $x + 4$ and $y - 3$ has the same area as the rectangle with side lengths x and y, it follows that

$$(x + 4)(y - 3) = xy. \qquad (1)$$

Similarly, because the rectangle with side lengths $x + 8$ and $y - 4$ has the same area as the rectangle with side lengths x and y,

$$(x + 8)(y - 4) = xy. \qquad (2)$$

By expanding the left hand sides of equations (1) and (2), we obtain

$$xy - 3x + 4y - 12 = xy \qquad (3)$$

and

$$xy - 4x + 8y - 32 = xy. \qquad (4)$$

By subtracting xy from both sides of equations (3) and (4), it follows that

$$-3x + 4y = 12 \qquad (5)$$

and

$$-4x + 8y = 32. \qquad (6)$$

We now subtract twice equation (5) from equation (6). This gives

$$(-4x + 8y) - (-6x + 8y) = 32 - 24.$$

That is, $2x = 8$, and hence

$$x = 4.$$

Substituting this value for x in equation (5) we deduce that

$$-12 + 4y = 12.$$

Hence

$$4y = 24$$

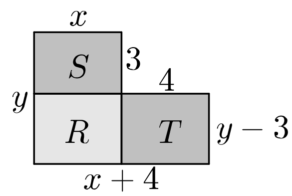
and it follows that

$$y = 6.$$

Therefore $x + y = 4 + 6 = 10$.

METHOD 2

In the figure on the right we have drawn the rectangle with side lengths $x + 4$ and $y - 3$ overlapping the rectangle with side lengths x and y. This produces three rectangles. We let their areas be R, S and T, as indicated in the figure.



As the two original rectangles have the same area, $R + S = R + T$, and therefore $S = T$.

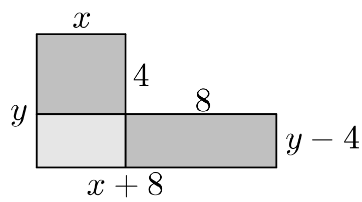
We see from the figure that S is the area of a rectangle with side lengths x and 3, and T is the area of a rectangle with side lengths 4 and $y - 3$. Therefore

$$3x = 4(y - 3).$$

This equation may be rearranged as

$$-3x + 4y = 12. \qquad (1)$$

Similarly, by drawing the rectangle with side lengths $x + 8$ and $y - 4$ overlapping the rectangle with side lengths x and y, as shown in the figure on the right, we obtain a rectangle with side lengths 4 and x, and a rectangle with side lengths 8 and $y - 4$ which have equal areas.



Therefore

$$4x = 8(y - 4),$$

and hence

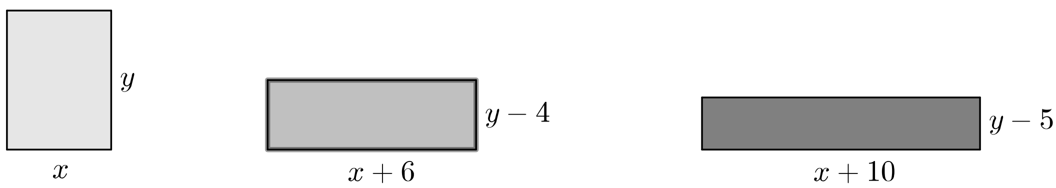
$$-4x + 8y = 32. \qquad (2)$$

Equations (1) and (2) are the same as equations (5) and (6) of Method 1. Therefore, as in Method 1, we may deduce from these equations that $x + y = 4 + 6 = 10$.

FOR INVESTIGATION

19.1 The three rectangles shown below all have the same area.

Find the values of x and y.



20. A particular integer is the smallest multiple of 72, each of whose digits is either 0 or 1.

How many digits does this integer have?

A 4 B 6 C 8 D 10 E 12

SOLUTION **E**

In this question we use the notation '$\dots cba$' for the positive integer which has units digit a, tens digit b, hundreds digit c, and so on.

We first note that $72 = 8 \times 9$. Hence, as 8 and 9 have no common factors (other than 1), the test for whether a number is divisible by 72 is that it should be divisible both by 8 and by 9.

Next we consider the criteria for divisibility by 8 and by 9. These are as follows.

- For the number '$\dots cba$' to be divisible by 8, the number 'cba' made up of its last three digits has to be divisible by 8.
- For the number '$\dots cba$' to be divisible by 9, the sum of its digits has to be divisible by 9.

It is easy to check that if each of the digits a, b and c is either 0 or 1, and 'cba' is divisible by 8, then 'cba' = '000'.

The sum of the digits of an integer each of whose digits is 0 or 1 is equal to the number of its digits that are 1s. So if the integer is to be divisible by 9, the number of its digits that are 1s has to be a multiple of 9. The fewer the number of digits, the smaller the number will be. The smallest positive multiple of 9 is, of course, 9 itself.

So the integer we seek is the integer with the smallest number of digits which ends with three 0s and includes nine 1s. This the integer which consists of nine 1s followed by three 0s.

Therefore the required integer is 111 111 111 000. This has 12 digits.

FOR INVESTIGATION

20.1 Check the truth of the statement in the above argument that if each of the digits a, b and c is either 0 or 1, and 'cba' is divisible by 8, then 'cba' = '000'.

20.2 Explain why the criterion given in the above argument for a number to be divisible by 8 is correct.

20.3 Explain why the criterion given in the above argument for a number to be divisible by 9 is correct.

20.4 Which is the smallest integer, each of whose digits is either 0 or 1, which is greater than 111 111 111 000 and is divisible by 72?

20.5 Factorize 111 111 111 000 into prime factors.

21. For certain values of x, the list x, $x + 6$ and x^2 contains just two different numbers.

How many such values of x are there?

A 1 B 2 C 3 D 4 E 5

Solution **D**

Whatever value x takes, x is not equal to $x + 6$. Therefore, the values of x for which there are just two different numbers in the list are those for which either $x^2 = x$ or $x^2 = x + 6$.

The equation

$$x^2 = x$$

may be rearranged as

$$x^2 - x = 0.$$

The left-hand side of this last equation may be factorized to give

$$x(x - 1) = 0.$$

From this we see that there are two values of x for which $x^2 = x$, namely 0 and 1.

The equation

$$x^2 = x + 6$$

may be rearranged as

$$x^2 - x - 6 = 0.$$

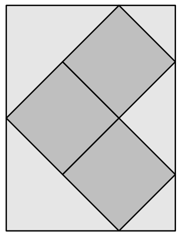
The left-hand side of this last equation may be factorized to give

$$(x + 2)(x - 3) = 0.$$

From this we see that there are two values of x for which $x^2 = x + 6$, namely -2 and 3.

Therefore there are altogether 4 different values for x for which there are just two different numbers in the list, namely, -2, 0, 1 and 3.

22. Three squares, with side-lengths 2, are placed together edge-to-edge to make an L-shape. The L-shape is placed inside a rectangle so that all five vertices of the L-shape lie on the rectangle, one of them at the midpoint of an edge, as shown.



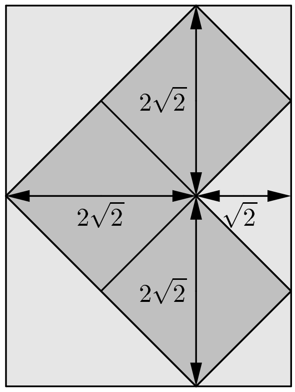
What is the area of the rectangle?

A 16 B 18 C 20 D 22 E 24

Solution **E**

Method 1

Let d be the length of the diagonals of the squares. Each diagonal is the hypotenuse of a right-angled triangle whose other sides are each of length 2. Therefore, by Pythagoras' Theorem $d^2 = 2^2 + 2^2 = 8$. Hence $d = \sqrt{8} = 2\sqrt{2}$.



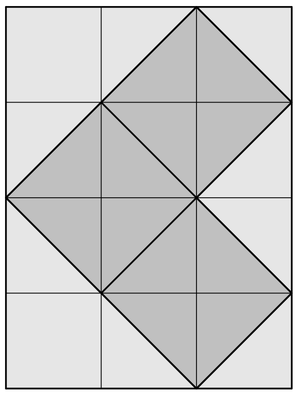
We therefore see from the figure that the width of the rectangle is $2\sqrt{2} + \sqrt{2}$, that is, $3\sqrt{2}$. Also, the height of the rectangle is $2\sqrt{2} + 2\sqrt{2}$, that is, $4\sqrt{2}$.

Hence the area of the rectangle is given by

$$3\sqrt{2} \times 4\sqrt{2} = 3 \times 4 \times \sqrt{2}^2 = 3 \times 4 \times 2 = 24.$$

Method 2

We divide the rectangle into 12 congruent smaller squares as shown in the figure.



From the figure we see that the three 2×2 squares cover 12 triangles each of which is half of one of 12 smaller squares.

It follows that the three larger squares cover half the area of the rectangle. Therefore the area of the rectangle is twice the area of the three 2×2 squares. Therefore the area of the rectangle is

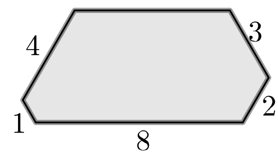
$$2 \times \big(3 \times (2 \times 2)\big) = 24.$$

For investigation

22.1 Both the above methods tacitly assume that the angles between the sides of the squares and the sides of the rectangle are all 45°.

Prove that this assumption is correct.

23. The diagram shows a hexagon. All the interior angles of the hexagon are 120°. The lengths of some of the sides are indicated.



What is the area of the hexagon?

A $20\sqrt{3}$ B $21\sqrt{3}$ C $22\sqrt{3}$ D $23\sqrt{3}$ E $24\sqrt{3}$

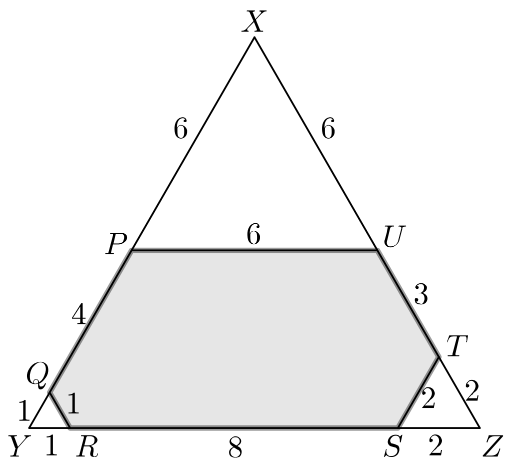
SOLUTION **A**

We give two methods. Both use the fact that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. The proof of this is left as an exercise (See Problem 23.1).

METHOD 1

The solution by Method 1 may seem rather long. But this is only because we have given the geometrical details. In the context of IMC, where you are not asked to explain your solutions, the answer could be obtained using this method quite quickly.

We have labelled the vertices of the hexagon as shown in the figure. The points where the sides of the hexagon QP, RS and TU meet when extended have been labelled X, Y and Z, as shown.



Because all the interior angles of the hexagon are 120° and the angles at a point have sum 180°, we have $\angle RQY = \angle QRY = 60°$. Because the sum of the angles in a triangle is 180°, it follows that $\angle RYQ = 60°$. Therefore the triangle QYR is equilateral, and hence $YR = YQ = QR = 1$.

Similarly, the triangle TSZ is equilateral and $SZ = ZT = ST = 2$.

Similarly, the triangle XPU is equilateral. We work out the length of its sides as follows.

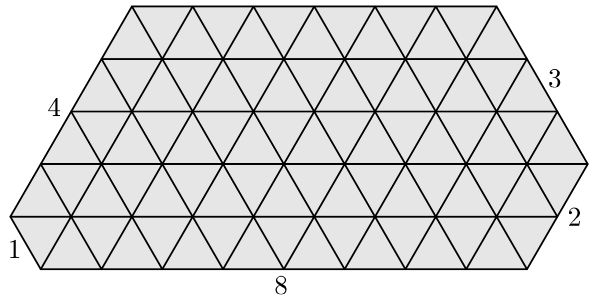
We see that YZ has length $1 + 8 + 2$, that is, 11. Also, because $\angle RYQ = \angle TZS = 60°$, the triangle XYZ is equilateral. Therefore $XY = XZ = YZ = 11$. It follows that $PX = 6$. Hence XU and PU also have length 6.

The area of the hexagon $PQRSTU$ is equal to the area of the equilateral triangle XYZ, less the areas of the equilateral triangles QYR, TSZ and XPU. Therefore, using the formula $\frac{\sqrt{3}}{4}s^2$ for the area of an equilateral triangle with side length s, we deduce that the area of the hexagon is given by

$$\begin{aligned}\frac{\sqrt{3}}{4}(11^2) - \frac{\sqrt{3}}{4}(1^2) - \frac{\sqrt{3}}{4}(2^2) - \frac{\sqrt{3}}{4}(6^2) &= \frac{\sqrt{3}}{4}(11^2 - 1^2 - 2^2 - 6^2)\\ &= \frac{\sqrt{3}}{4}(121 - 1 - 4 - 36) = \frac{\sqrt{3}}{4}(80) = 20\sqrt{3}.\end{aligned}$$

METHOD 2

We divide the hexagon into equilateral triangles all of which have side length 1, as shown in the figure. By counting we see that there are 80 of these triangles.



It follows from the formula $\frac{\sqrt{3}}{4}s^2$ for the area of an equilateral triangle with side length s that each of 80 equilateral triangles with side length 1 has has area $\frac{\sqrt{3}}{4}$.

Therefore the total area of these triangles is

$$80 \times \frac{\sqrt{3}}{4} = 20\sqrt{3}.$$

FOR INVESTIGATION

23.1 Prove that the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$.

[A proof using the formula $\frac{1}{2}ab\sin\theta$ for the area of a triangle with two sides with lengths a and b and included angle θ is complete only if you show that this formula is correct, and that $\sin 60° = \frac{\sqrt{3}}{2}$.]

23.2 The hexagon of this question is a hexagon all of whose interior angles are 120° but all of whose sides have different lengths. Give another example of a hexagon with these properties.

23.3 Find a condition on a sequence a, b, c, d, e, f of positive integers for there to be a hexagon all of whose angles are 120° and whose side lengths are a, b, c, d, e and f as you go round the hexagon anticlockwise.

[Note that the existence of regular hexagons shows that, for each positive integer k, the sequence k, k, k, k, k, k has the required property. Also, the hexagon of this question shows that the sequence 1, 8, 2, 3, 6, 4 has this property.]

24. A list of 5 positive integers has mean 5, mode 5, median 5 and range 5.

How many such lists of 5 positive integers are there?

A 1 B 2 C 3 D 4 E 5

SOLUTION **B**

We suppose that the positive integers in the list are a, b, c, d and e, where $a \leq b \leq c \leq d \leq e$.

Because the median is 5, we have $c = 5$.

Because the mode is 5, there must be at least two 5s in the list. Therefore, either $b = 5$ or $d = 5$.

Because the range is 5, we have $e = a + 5$. Hence the list is either

$$a,\ 5,\ 5,\ d,\ a+5 \quad \text{or} \quad a,\ b,\ 5,\ 5, a+5.$$

Because the 5 numbers in the list have mean 5, their sum is 5×5, that is, 25. Therefore in the first case $a + 5 + 5 + d + (a + 5) = 25$ and hence $d = 10 - 2a$, and in the second case, $a + b + 5 + 5 + (a + 5) = 25$ and hence $b = 10 - 2a$.

Hence the list is either

$$a,\ 5,\ 5,\ 10-2a,\ a+5 \quad \text{or} \quad a,\ 10-2a,\ 5,\ 5, a+5.$$

In the first case we have $5 \leq 10 - 2a \leq a + 5$, and hence $2a \leq 5$ and $5 \leq 3a$. It follows that, as a is an integer, 2 is the only possible value for a.

In the second case we have $a \leq 10 - 2a \leq 5$, and hence $3a \leq 10$ and $5 \leq 2a$. It follows that, as a is an integer, 3 is the only possible value for a.

It follows that the only lists that satisfy the given conditions are

$$2,\ 5,\ 5,\ 6,\ 7 \quad \text{and} \quad 3,\ 4,\ 5,\ 5,\ 8.$$

Therefore there are just 2 lists that meet the given conditions.

FOR INVESTIGATION

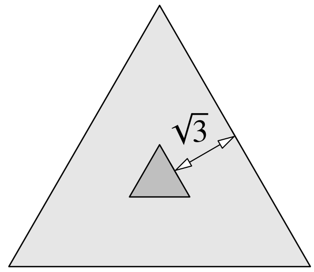
24.1 How many lists are there of 6 positive integers which have mean 6, mode 6, median 6 and range 6?

24.2 How many lists are there of 7 positive integers which have mean 7, mode 7, median 7 and range 7?

25. The diagram shows two equilateral triangles. The distance from each point of the smaller triangle to the nearest point of the larger triangle is $\sqrt{3}$, as shown.

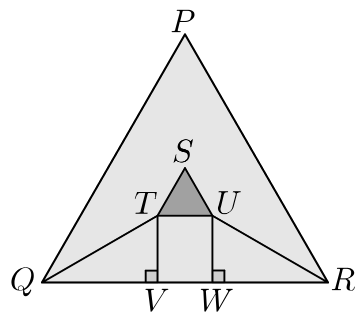
What is the difference between the lengths of the edges of the two triangles?

A $2\sqrt{3}$ B $4\frac{1}{2}$ C $3\sqrt{3}$ D 6 E $4\sqrt{3}$



SOLUTION **D**

We label the vertices of the two triangles as shown in the figure. We let V and W, respectively, be the points where the perpendiculars from T and U to QR meet QR.



Because each point of the triangle STU is at the distance $\sqrt{3}$ from the nearest point of the triangle PQR, we have $TV = UW = \sqrt{3}$.

Similarly, the distance from T to PQ is $\sqrt{3}$. Therefore T is equidistant from the sides QR and QP of the triangle PQR.

Hence the line QT bisects the angle of the equilateral triangle at Q. [You are asked to prove this in Problem 25.1.] Therefore $\angle VQT = 30°$.

It follows that the triangle TQV has one right angle and its other angles are 30° and 60°. Therefore $QT = 2TV = 2\sqrt{3}$.

By Pythagoras' Theorem applied to this triangle, $QT^2 = QV^2 + TV^2$. It follows that $QV^2 = QT^2 - TV^2 = (2\sqrt{3})^2 - \sqrt{3}^2 = 12 - 3 = 9$. Hence $QV = 3$. Similarly $WR = 3$.

Because $TVWU$ is a rectangle, $TU = VW$. Hence, we have

$$QR = QV + VW + WR = 3 + TU + 3 = TU + 6.$$

It follows that the difference between the lengths of the edges of the two triangles is 6.

FOR INVESTIGATION

25.1 Prove that, because T is equidistant from the sides QR and QP of the triangle PQR, it follows that QT bisects the angle of the triangle at Q.

25.2 Explain why it follows from the facts that $\angle QVT = 90°$ and $\angle TQV = 30°$ that $QT = 2TV$.

25.3 Prove that $TVWU$ is a rectangle.