EXAMINATION

29 April 2009 (am)

Subject CT4 — Models
Core Technical

Time allowed: Three hours

INSTRUCTIONS TO THE CANDIDATE

1. *Enter all the candidate and examination details as requested on the front of your answer booklet.*

2. *You must not start writing your answers in the booklet until instructed to do so by the supervisor.*

3. *Mark allocations are shown in brackets.*

4. *Attempt all 12 questions, beginning your answer to each question on a separate sheet.*

5. *Candidates should show calculations where this is appropriate.*

Graph paper is not required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.

1 A life insurance company has a small group of policies written on impaired lives and has conducted an investigation into the mortality of these policyholders. It is proposed that the crude mortality rates be graduated for use in future premium calculations.

Discuss the suitability of two methods of graduation that the insurance company could use. [3]

2 (i) Explain what is meant by a time-homogeneous Markov chain. [2]

Consider the time-homogeneous two-state Markov chain with transition matrix:

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

(ii) Explain the range of values that a and b can take which result in this being a valid Markov chain which is:

(a) irreducible
(b) periodic [3]

[Total 5]

3 List the benefits and limitations of modelling in actuarial work. [5]

4 Below is an extract from English Life Table 15 (females).

Age x (years)	*Number of survivors to exact age x out of 100,000 births*
30	98,617
40	97,952

(i) Calculate ${}_5q_{30}$ under each of the two following alternative assumptions:

(a) a uniform distribution of deaths (UDD) between ages 30 and 40 years
(b) a constant force of mortality between ages 30 and 40 years [3]

(ii) Calculate the number of survivors to exact age 35 years out of 100,000 births under each of the assumptions in (i) above. [1]

English Life Table 15 (females) was originally calculated using data classified by single years of age. The number of survivors to exact age 35 years was 98,359.

(iii) Comment on the appropriateness of the assumptions of UDD and a constant force of mortality between ages 30 and 40 years in this example. [3]

[Total 7]

5 Explain the basis underlying the grouping of signs test, and derive the formula for the probability of exactly t positive groups by considering the possible arrangements of a set of positive and negative signs. [5]

6 An investigation by a hospital into rates of recovery after a specific type of operation collected the following data for each month of the calendar year 2008:

- number of persons who recovered from the operation during the month (defined as being discharged from the hospital) classified by the month of their operation.

You may assume that there were no deaths.

On the first day of each month from January 2008 to January 2009, the hospital listed all in-patients who were yet to recover from this operation, classified according to the length of time elapsing since their operation, to the nearest month.

(i) (a) Write down an expression which will enable the hospital to calculate rates of recovery, r_x, during 2008 at various durations x since the operation using the available data.

(b) Derive a formula for the exposed to risk based on the information in the hospital's monthly lists of in-patients which corresponds to the data on recovery from the operation.

[5]

(ii) Determine the value of f such that the expression in (i)(a) applies to an actual duration $x + f$ since the operation. [2]

[Total 7]

7 (i) Explain how the classification of stochastic processes according to the nature of their state space and time space leads to a four way classification. [2]

(ii) For each of the four types of process:

(a) give an example of a statistical model

(b) write down a problem of relevance to the operation of:

- a food retailer
- a general insurance company

[6]

[Total 8]

8 There is a population of ten cats in a certain neighbourhood. Whenever a cat which has fleas meets a cat without fleas, there is a 50% probability that some of the fleas transfer to the other cat such that both cats harbour fleas thereafter. Contacts between two of the neighbourhood cats occur according to a Poisson process with rate μ, and these meetings are equally likely to involve any of the possible pairs of individuals. Assume that once infected a cat continues to have fleas, and that none of the cats' owners has taken any preventative measures.

(i) If the number of cats currently infected is x, explain why the number of possible pairings of cats which could result in a new flea infection is $x(10 - x)$. [1]

(ii) Show how the number of infected cats at any time, $X(t)$, can be formulated as a Markov jump process, specifying:

(a) the state space
(b) the Kolmogorov differential equations in matrix form

[4]

(iii) State the distribution of the holding times of the Markov jump process. [2]

(iv) Calculate the expected time until all the cats have fleas, starting from a single flea-infected cat. [2]
[Total 9]

9 (i) Prove that, under Gompertz's Law, the probability of survival from age x to age $x + t$, ${}_t p_x$, is given by:

$$ {}_t p_x = \left[\exp\left(\frac{-B}{\ln c} \right) \right]^{c^x (c^t - 1)} . \quad [3] $$

For a certain population, estimates of survival probabilities are available as follows:

$$ {}_1 p_{50} = 0.995 $$
$$ {}_2 p_{50} = 0.989 . $$

(ii) Calculate values of B and c consistent with these observations. [3]

(iii) Comment on the calculation performed in (ii) compared with the usual process for estimating the parameters from a set of crude mortality rates. [3]
[Total 9]

10 Let T_x be a random variable denoting future lifetime after age x, and let T be another random variable denoting the lifetime of a new-born person.

(i) (a) Define, in terms of probabilities, $S_x(t)$, which represents the survival function of T_x.

(b) Derive an expression relating $S_x(t)$ to $S(t)$, the survival function of T.

[2]

(ii) Define, in terms of probabilities involving T_x, the force of mortality, μ_{x+t}.

[1]

The Weibull distribution has a survival function given by

$$S_x(t) = \exp\left(-(\lambda t)^{\beta}\right),$$

where λ and β are parameters ($\lambda, \beta > 0$).

(iii) Derive an expression for the Weibull force of mortality in terms of λ and β.

[3]

(iv) Sketch, on the same graph, the Weibull force of mortality for $0 \le t \le 5$ for the following pairs of values of λ and β:

$\lambda = 1, \ \beta = 0.5$
$\lambda = 1, \ \beta = 1.0$
$\lambda = 1, \ \beta = 1.5$

[4]
[Total 10]

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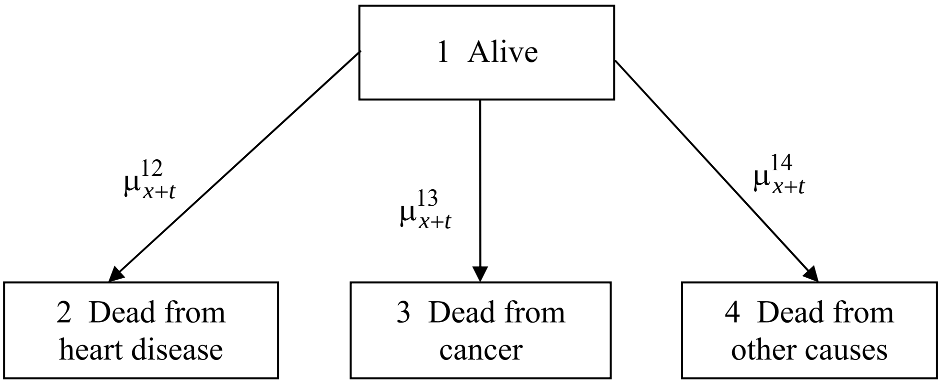
11 An investigation into mortality by cause of death used the four-state Markov model shown below.



(i) Show from first principles that

$$\frac{\partial}{\partial t}\,{}_t p_x^{12} = \mu_{x+t}^{12}\,{}_t p_x^{11}\,. \qquad [5]$$

The investigation was carried out separately for each year of age, and the transition intensities were assumed to be constant within each single year of age.

(ii) (a) Write down, defining all the terms you use, the likelihood for the transition intensities.

(b) Derive the maximum likelihood estimator of the force of mortality from heart disease for any single year of age. [5]

The investigation produced the following data for persons aged 64 last birthday:

Total waiting time in the state Alive	1,065 person-years
Number of deaths from heart disease	34
Number of deaths from cancer	36
Number of deaths from other causes	42

(iii) (a) Calculate the maximum likelihood estimate (MLE) of the force of mortality from heart disease at age 64 last birthday.

(b) Estimate an approximate 95% confidence interval for the MLE of the force of mortality from heart disease at age 64 last birthday. [3]

(iv) Discuss how you might use this model to analyse the impact of risk factors on the death rate from heart disease and suggest, giving reasons, a suitable alternative model. [3]

[Total 16]

12 A motor insurer operates a no claims discount system with the following levels of discount {0%, 25%, 50%, 60%}.

The rules governing a policyholder's discount level, based upon the number of claims made in the previous year, are as follows:

- Following a year with no claims, the policyholder moves up one discount level, or remains at the 60% level.
- Following a year with one claim, the policyholder moves down one discount level, or remains at 0% level.
- Following a year with two or more claims, the policyholder moves down two discount levels (subject to a limit of the 0% discount level).

The number of claims made by a policyholder in a year is assumed to follow a Poisson distribution with mean 0.30.

(i) Determine the transition matrix for the no claims discount system. [3]

(ii) Calculate the stationary distribution of the system, π. [5]

(iii) Calculate the expected average long term level of discount. [1]

The following data shows the number of the insurer's 130,200 policyholders in the portfolio classified by the number of claims each policyholder made in the last year. This information was used to estimate the mean of 0.30.

No claims	96,632
One claim	28,648
Two claims	4,400
Three claims	476
Four claims	36
Five claims	8

(iv) Test the goodness of fit of these data to a Poisson distribution with mean 0.30. [5]

(v) Comment on the implications of your conclusion in (iv) for the average level of discount applied. [2]

[Total 16]

END OF PAPER