Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

April 2009

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

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Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

There were some excellent performances and well-prepared candidates scored well across the whole paper. However, the comments below on each question concentrate on areas where candidates could have improved their performance.

Q1, Q2.

As has often been the case when words rather than numbers have been required, these bookwork questions were answered relatively poorly (although Q2 was answered better than Q1).

Q3.

Well answered.

Q4.

Defining an arbitrage profit correctly was also acceptable as an answer to (i) although a description of both possible arbitrage scenarios was required for full marks. Many candidates performed the calculations well although the methodology being used was not always clear.

Q5.

The question required an ability to bring together two separate elements of the syllabus and less well-prepared candidates seemed to struggle with this.

Q6.

This was another question where students scored relatively poorly with many candidates having difficulty with the income calculation. A common error was to assume that the income rose by 4% every three years.

Q7.

This was answered much better than questions on the same topic in previous exams. However, some candidates did confuse the money-weighted and time-weighted rates of return.

Q8.

It was particularly disappointing to see many candidates using the wrong formula for DMT in part (i) but ending their proof with'=14.42 QED' in the final line. This suggests a lack of professionalism, honesty and integrity which are key attributes of the actuarial profession.

Part (ii) was well-answered with various different methods leading to the correct answer.

Q9.

This was the worst-answered question on the paper although it was still possible to score significant marks by calculating forward rates using the correct formula even if the spot rates had been calculated incorrectly.

Q10.

Part (i) was answered well but many candidates lost marks in part (ii) by not realising that a separate test was required to ascertain the worst time to redemption. Many candidates calculated the annual effective yield rather than the yield per annum convertible quarterly in part (iii).

Q11.

Many candidates seemed confused as to what to calculate in part (i) and failed to distinguish between the premium needed in 10 years' time and the present value of that premium. Part (ii) was answered well (although some candidates appeared to be short of time at this stage). Part (iii) was answered very poorly with many candidates not appreciating the effects of the high variance.

1 Characteristics of government bills:

- short-dated securities issued by governments to fund their short-term spending requirements.
- issued at a discount and redeemed at par with no coupon.
- mostly denominated in the domestic currency, although issues can be made in other currencies.
- yield is typically quoted as a simple rate of discount for the term of the bill
- absolutely secure
- often highly marketable despite being unquoted.
- often used as a benchmark risk-free short-term investment.

2 (a) An interest-only loan requires the borrower only to pay interest on the entire loan in each time period. The loan does not reduce over time so the interest remains constant. A separate investment or savings account can be established in which payments are made to extinguish the whole loan at the end of the term.

(b) A repayment loan involves level repayments of capital and interest. The first part of the payment is used to pay interest on any remaining capital. The remaining part of the payment is then used to repay capital so that the capital gradually reduces over the term of the loan.

3 (i) $300a_{\overline{20}|} + 30v\left(Ia\right)_{\overline{19}|}$ at 7%

$= 300\left(10.594\right) + 30 \times \frac{1}{1.07} \times 82.9347 = 5503.47$

(ii) Capital outstanding after 5 payments:

$420a_{\overline{15}|} + 30\left(Ia\right)_{\overline{15}|}$

$= 420 \times 9.1079 + 30 \times 61.5540 = 5671.94$

(iii) Cap o/s after 19 payments = 870v @ 7% = £813.08

= Capital in the final payment

Interest in the final payment = 870 – 813.08 = £56.92

4 (i) The "no arbitrage" assumption means that **neither** of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The forward price at the outset of the contract was:

$$\left(94.5 - 9v_{5\%}^{10} - 10v_{5\%}^{11}\right) \times (1.05)^{12} = 149.29$$

The forward price that should be offered now is:

$$\left(143 - 9v_{5\%}^{2} - 10v_{5\%}^{3}\right) \times (1.05)^{4} = 153.39$$

Hence the value of the contract now is:

$$(153.39 - 149.29)\, v_{5\%}^{4} = 3.37$$

Note:

This result can also be obtained directly from:

$$143 - 94.5 \times (1.05)^{8} = 3.38$$

since the coupons are irrelevant in this calculation.

5 Working in £000's

PV of outgo $= 100 + 80e^{-\int_0^2 (0.05+0.002t)\,dt}$

$= 100 + 80e^{-\left[0.05t+0.001t^2\right]_0^2}$

$= 100 + 80e^{-0.104} = 172.10$

DPP is value of T for which:

PV (income paid up to T) = PV (outgo)

Where

PV (income paid up to T) = $\int_8^T 100e^{0.001t}\ v(t)\,dt$

and $v(t) = e^{-\left[\int_0^5 (0.05+0.002t)dt + \int_5^t 0.06 dt\right]}$

$= e^{-\left[0.05t+0.001t^2\right]_0^5} . e^{-(0.06t-0.30)}$

$= e^{-0.275} . e^{-0.06t}\ e^{0.30}$

$= e^{0.025}\ e^{-0.06t}$

$\Rightarrow PV(\text{income paid up to } T) = \int_8^T 100e^{0.001t}\ e^{0.025}\ e^{-0.06t} dt$

$= \int_8^T 100e^{0.025}\ e^{-0.059t} dt$

$= \frac{100}{-0.059} e^{0.025}\left[e^{-0.059T} - e^{-0.059\times 8}\right]$

$= -1737.8222\ e^{-0.059T} + 1083.97$

$\Rightarrow$ DPP is T such that

$172.10 = -1737.8222 e^{-0.059T} + 1083.97$

$\Rightarrow e^{-0.059T} = 0.52472$

$\Rightarrow -0.059T = Ln(0.52472) \Rightarrow T = 10.93$ years

6 Working in 000's

PV of costs $= 5000 + 900v^{11/12}$ at 8%

$= 5838.695$

PV of income $= 800v^{13/12}\left(\ddot{a}_{\overline{3}|}^{(4)} + 1.04^3 v^3 \ddot{a}_{\overline{3}|}^{(4)} + \cdots + (1.04)^{12} v^{12}\ \ddot{a}_{\overline{3}|}^{(4)}\right)$

$= 800v^{13/12}\ \ddot{a}_{\overline{3}|}^{(4)}\left(1 + (1.04v)^3 + \cdots (1.04v)^{12}\right)$

$= 800 \times 0.908281 \times 1.049519 \times 2.5771 \times \left(\dfrac{1-\left(\frac{1.04}{1.08}\right)^{15}}{1-\left(\frac{1.04}{1.08}\right)^{3}}\right)$

$= 1965.3133 \times 4.038121$

$= 7936.173$

PV of proceeds from sale $= 6000v^{16\,3/12} = 1717.969$

NPV of project = 7936.173+1717.969 – 5838.695

= 3815.447 (i.e. £3,815,447)

7 Working in 000's

(i) TWRR is *i* such that

$$(1+i)^{2\frac{1}{2}} = \frac{175}{150} \times \frac{225}{175+30} \times \frac{280}{225+40}$$

$$= \frac{175}{150} \times \frac{225}{205} \times \frac{280}{265} = 1.352968$$

$\therefore\quad i = 12.85\%$ p.a.

(ii) MWRR is i such that

$$150(1+i)^{2\frac{1}{2}} + 30(1+i)^{1\frac{1}{2}} + 40(1+i)^{\frac{1}{2}} = 280$$

Try: i = 12%, LHS = 277.02
i = 12.5%, LHS = 279.58
i = 13%, LHS = 282.16

$$\therefore i = 12.5\% + \frac{(28 - 27.958)}{(28.216 - 27.958)} \times 0.5\%$$

= 12.58% p.a.

(iii) The TWRR is better for comparing 2 investment manager's performances as it is not sensitive to cash flow amounts and timing of payments. The MWRR is sensitive to both.

8 (i) Working in £m

Discounted mean term =

$$\frac{10v^{10} + 11v^{11} + 12v^{12} + \ldots\ldots\ldots\ldots + 20v^{20}}{v^{10} + v^{11} + v^{12} + \ldots\ldots\ldots\ldots + v^{20}}$$

$$= \frac{10v + 11v^2 + 12v^3 + \ldots\ldots\ldots + 20v^{11}}{v + v^2 + v^3 + \ldots\ldots\ldots + v^{11}}$$

$$= \frac{9a_{\overline{11}|} + (Ia)_{\overline{11}|}}{a_{\overline{11}|}} = 9 + \frac{(Ia)_{\overline{11}|}}{a_{\overline{11}|}} \quad \text{at 6\%}$$

$(Ia)_{\overline{11}|} = 42.7571$

$\Rightarrow DMT = \quad 9 + \dfrac{42.7571}{7.8869} = 14.42128$

$\Rightarrow$ to 4 significent figures DMT = 14.42

(ii) First condition: pv assets = pv liabilities

$$\Rightarrow \quad Xv^{10} + Yv^{20} = v^9 a_{\overline{11}|} * 1 \qquad \text{at 6\%.}$$

$X * 0.55839 + Y * 0.31180 = 0.59190 * 7.8869$ (using tables)

= 4.668256 ………….(1)

2nd condition: DMT assets = DMT liabilities

$\Rightarrow \dfrac{X*10v^{10}+Y*20v^{20}}{Xv^{10}+Yv^{20}}$ = 14.42128 (use of 14.42 from (i) will be accepted)

$\Rightarrow X*5.5839+Y*6.236$ = $14.42128*\left(Xv^{10}+Yv^{20}\right)$

= $14.42128*4.668256$ from (1)

= 67.3222 (or 67.3163 if DMT of 14.42 is used)…………(2)

Equ^{n} (2) – 10* Equ^{n} (1) $\Rightarrow$

$Y*6.236-Y*3.1180=67.3222-10*4.668256$

$\Rightarrow Y=\dfrac{20.639667}{3.1180}=6.6195$ (or 6.6176 if DMT of 14.42 is used)

[or $V_A^{'}=V_L^{'}$ (differentiating with respect to *i*)

$$10Xv^{11}+20Yv^{21}=10v^{11}+11v^{12}+\ldots+20v^{21}$$
$$=v^{10}\left(9a_{\overline{11|}}+\left(Ia\right)_{\overline{11|}}\right)$$

$\Rightarrow 5.2679X+5.8831Y$ = 63.5112 …………(2)

Equ^{n} (2) – $\dfrac{5.2679}{5.8831}\times$ Equ^{n} (1)

$\Rightarrow 2.94155Y=19.4711\Rightarrow Y=6.6193$]

Equ^{n} (1) $\Rightarrow$ X * 0.55839 = 4.668256 – 6.6195 * 0.31180

$\Rightarrow$ X = 4.6639 (or 4.6650 if DMT of 14.42 is used)

[check, in equ^{n} (2). 4.6639 * 5.5839 + 6.6195 * 6.236 = 67.3222]

(iii) For the third condition to be satisfied, it is necessary for the spread of the assets to exceed the spread of the liabilities. This appears to be the case given that the liabilities occur in equal annual amounts at durations from 10 years to 20 years, whereas the assets are concentrated in two lumps at the two most extreme durations, 10 years and 20 years.

9 Let the 1-year and 2-year zero-coupon yields (spot rates) be i_i and i_2 respectively.

$$\frac{105}{1+i_1} = 105v \text{ @ } 4.5\%$$

$$\therefore i_1 = 0.045$$

For the 2-year spot rate:

$$\frac{5}{1+i_1} + \frac{105}{(1+i_2)^2} = 5a_{\overline{2}|5.3\%} + 100v^2_{5.3\%}$$

$$\frac{5}{1.045} + \frac{105}{(1+i_2)^2} = 5\frac{\left(1-\frac{1}{1.053^2}\right)}{0.053} + \frac{100}{1.053^2}$$

= 9.257681 + 90.186858

= 99.444539

$$\frac{105}{(1+i_2)^2} = 99.444539 - \frac{5}{1.045}$$

$$\Rightarrow (1+i_2)^2 = \frac{105}{94.659850}$$

$$\Rightarrow i_2 = 5.3202\% \text{ p.a.}$$

For the 3-year spot rate:

The 3-year par yield is 5.6% p.a.

$$\Rightarrow 1 = 0.056\left(\frac{1}{1+i_1} + \frac{1}{(1+i_2)^2} + \frac{1}{(1+i_3)^3}\right) + \frac{1}{(1+i_3)^3}$$

$$\Rightarrow \frac{1.056}{(1+i_3)^3} = 1 - \frac{0.056}{1.045} - \frac{0.056}{(1.053202)^2}$$

$$\Rightarrow (1+i_3)^3 = \frac{1.056}{0.895926}$$

$$\Rightarrow i_3 = 5.6324\% \text{ p.a.}$$

1-year forward rates:

$f_0 = i_1 = 4.5\%$ p.a.

$$(1+i_1)(1+f_1) = (1+i_2)^2$$

$$\Rightarrow 1+f_1 = \frac{1.053202^2}{1.045}$$

$\Rightarrow f_1 = 6.1468\%$ p.a.

$$(1+i_2)^2(1+f_2) = (1+i_3)^3$$

$$\Rightarrow 1+f_2 = \frac{(1.056324)^3}{(1.053202)^2}$$

$\Rightarrow f_2 = 6.2596\%$ p.a.

10 (i) check for capital gain:

$$g(1-t_1) = \frac{0.11}{1.15} * (1-0.3)$$

$= 0.06696$

$i = 8\% \Rightarrow i^{(4)} = 0.077706$

$\Rightarrow i^{(4)} > g(1-t_1)$

$\Rightarrow$ There's a capital gain and thus loan should be assumed to be redeemed at the latest possible date.

Let P be price at which the investor bought the loan.

Then

$$P = 11 \times 0.7 a_{\overline{15|}}^{(4)} + 115v^{15} - 0.25(115-P)v^{15} \text{ at } 8\%$$

$$\Rightarrow P = \frac{7.7 \times 1.029519 \times 8.5595 + 0.75 \times 115 \times 0.31524}{1 - 0.25 \times 0.31524}$$

= £103.17 per £100 nominal

(ii) check for capital gain:

$$g\left(1-t_1\right)=\frac{0.11}{1.15}=0.095652$$

$i=9\% \Rightarrow i^{(4)}=0.087113$

$\Rightarrow i^{(4)}<g\left(1-t_1\right)$

$\Rightarrow$ There's no capital gain and thus loan should be assumed to be redeemed at the earliest possible date.

Let P' be the price at which the investor sold the loan. Then

$$P'=11a_{\overline{7|}}^{(4)}+115v^7 \text{ at } 9\%$$

$$=11\times 1.033144\times 5.033+115\times 0.54703$$

$$=\text{£}120.1064 \text{ per £}100 \text{ nominal}$$

(iii) Let j be the yield per quarter. Then

$$103.17=\frac{11}{4}\times 0.7a_{\overline{12|}}+120.1064v^{12}-0.25\left(120.1064-103.17\right)v^{12} \text{ at } j\ \%$$

$$\Rightarrow 103.17=1.925\,a_{\overline{12|}}+115.8723\,v^{12}$$

Try

j = 3%: RHS = 100.4319638

j = 2.5%: RHS = 105.9042724

Linear interpolation:

$$j=0.025+0.005\times\frac{\left(103.17-105.9042724\right)}{\left(100.4319638-105.9042724\right)}$$

= 0.02749828

Hence, net yield is 11% p.a. (or 10.99931% p.a.) payable quarterly.

11 (i) In 10 years' time the single premium P is

$$P = 12000\left(a_{\overline{1}|}^{(12)} + 1.03a_{\overline{1}|}^{(12)}v + (1.03)^2 a_{\overline{1}|}^{(12)}v^2 + \ldots + (1.03)^{14} v^{14}a_{\overline{1}|}^{(12)}\right)$$

$$= 12000a_{\overline{1}|}^{(12)}\left(1 + \frac{1.03}{1.06} + \left(\frac{1.03}{1.06}\right)^2 + \ldots + \left(\frac{1.03}{1.06}\right)^{14}\right)$$

$$= 12000a_{\overline{1}|}^{(12)}\left(\frac{1 - \left(\frac{1.03}{1.06}\right)^{15}}{1 - \frac{1.03}{1.06}}\right)$$

where $a_{\overline{1}|}^{(12)} = \frac{i}{i^{(12)}}v$

$$= \frac{1.027211}{1.06} = 0.969067$$

$$\Rightarrow P = 12000 \times 0.969067 \times \frac{0.3499146}{0.0283019}$$

$$= 143{,}774.45$$

(ii) $E(1+i_t) = 1.06 = e^{\mu + \sigma^2/2}$

$$Var(1+i_t) = (0.15)^2 = e^{2\mu+\sigma^2}.\left(e^{\sigma^2} - 1\right)$$

Then $\frac{0.15^2}{(1.06)^2} = e^{\sigma^2} - 1$

$$\Rightarrow \sigma^2 = 0.01982706$$

$$\therefore \mu = \ell n\, 1.06 - \frac{0.01982706}{2}$$

$$= 0.04835538$$

$$\Rightarrow S_{10} \sim LN(0.4835538, 0.1982706)$$

Let X be the amount to be invested at time 0

We want $\Pr\left(X.S_{10} \geq 143,774.45\right) = 0.98$

so $\Pr\left(S_{10} \geq \dfrac{143,774.45}{X}\right) = 0.98$

so $1 - \Phi\left(\dfrac{Ln\frac{143774.45}{X} - 10\mu}{\sqrt{10\sigma^2}}\right) = 0.02$

$\Rightarrow \dfrac{Ln\frac{143774.45}{X} - 10\mu}{\sqrt{10\sigma^2}} = -2.0537$

So $Ln\dfrac{143774.45}{X} = -2.0537 \times \sqrt{0.1982706} + 0.4835538$

$= -0.430909$

$\Rightarrow \dfrac{143774.45}{X} = 0.6499179$

$\Rightarrow \quad X = £221,219.41$

(iii) It might seem odd that the initial investment needs to be substantially higher than the single premium required in 10 years' time to have a 98% probability of accumulating to the single premium.

This strange result is explained by the fact that the variance of the interest rate is so high relative to the mean. There is therefore a significant risk that the investment will decrease in value over the next 10 years.

END OF EXAMINERS' REPORT