Subject: Numerical Computing

Code: C-09/T-09 (June 2003)

1. (a) Let $x=\sqrt{N}$ or $x^2 = N$.

We take $f(x)=x^2-N$, $f'(x)=2x$. Newton-Raphson method becomes

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^2-N}{2x_n}=\frac{x_n^2+N}{2x_n}=\frac{1}{2}\left[x_n+\frac{N}{x_n}\right]$$ **Answer: A**

(b) $f[x_0,x_1]=[f(x_1)-f(x_0)]/(x_1-x_0)=\Delta f_0/h$

$f[x_1,x_2]=[f(x_2)-f(x_1)]/(x_2-x_1)=\Delta f_1/h$

$f[x_0,x_1,x_2]=[f[x_1,x_2]-f[x_0,x_1]]/(x_2-x_0)$

$=[\Delta f_1-\Delta f_0]/(2h^2)=\Delta^2 f_0/(2h^2)$ **Answer: C**

(c) Iteration matrix associated with the Gauss-Jacobi iteration method is

$\mathbf{H} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})$. From the given matrix **A**, we get

$$\mathbf{H}=-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 4\\ 2 & 0\end{bmatrix}=\begin{bmatrix}0 & -4\\ -2 & 0\end{bmatrix}$$

Eigen values of **H** are $\pm 2\sqrt{2}$.

Since the spectral radius of **H** is $2\sqrt{2}>1$, the method diverges. **Answer: D**

(d) We first construct the forward difference table from the given data. We have

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
1	-1			
2	-1	0		
3	1	2	2	
4	5	4	2	0

Using Newton forward difference interpolation and the given data, we get for $h = 1$

$$p(x)=-1+(x-1)(0)+\frac{(x-1)(x-2)}{2}(2)=x^2-3x+1$$ **Answer: B**

(e) For the Simpsons rule $\int_{x_0}^{x_2} f(x)dx = h[f(x_0)+4f(x_1)+f(x_2)]/3$

we obtain $h = 1/2$, $x_0 = 0$, $x_2 = 1$, $x_1 = 1/2$, $f_0 = 1$, $f_1 = 4/5$, $f_2 = 1/2$

$I = [1 + 4\,(4/5) + 1/2]/6 = 47/60$ **Answer: D**

(f) We need the approximation $f(x) = ax + b$, where a and b are to be determined so that

$I(a,b)=\sum_{i=1}^{4}[f(x_i)-ax_i-b]^2=$ minimum. We obtain the normal equations

$$\frac{\partial I}{\partial a} = -2\sum_{i=1}^{4}\left(f(x_i) - ax_i - b\right)x_i = 0 \quad \text{or} \quad \sum x_i f(x_i) - a\sum x_i^2 - b\sum x_i = 0$$

$$\frac{\partial I}{\partial b} = -2\sum_{i=1}^{4}\left(f(x_i) - ax_i - b\right) = 0 \quad \text{or} \quad \sum f(x_i) - a\sum x_i - 4b = 0$$

We have $\sum x_i = 10, \sum f(x_i) = 8, \sum x_i^2 = 30, \sum x_i f(x_i) = 30$

Substituting in the normal equations, we get

$30a + 10b = 30, \; 10a + 4b = 8$

Solving, we get $a = 2$, $b = -3$ **Answer : A**

(g) The truncation error associated with the given method is given by

$$TE = f'(x) - [f(x+h) + af(x) - f(x-h)]/(2h)$$
$$= f'(x) - [af(x) + 2hf'(x) + h^3 f'''(x)/3 + \ldots\ldots\ldots\ldots]/(2h)$$
$$= -af(x)/2h - h^2 f'''(x)/6 + \ldots\ldots\ldots$$

The method will be of the highest order, if the coefficient of $f(x)$ is zero.
We get $a = 0$. **Answer: B**

(h) Make the method exact for $f(x) = 1$ and x. We get

$f(x) = 1: \quad \int_{-1}^{1} dx = 1/2 + 3/2 \qquad \Rightarrow 2 = 2$ which is true

$f(x) = x: \quad \int_{-1}^{1} x\,dx = -1/2 + 3a/2 \quad \Rightarrow 0 = 1/2(3a - 1) \quad \Rightarrow a = 1/3$.

Answer: C

2. (a) We have $f(x) = x^2 - 2x - 3\cos x$. We find that $f(-2) = 9.2484$, $f(-1) = 1.3791$, $f(0) = -3$, $f(1) = -2.6209$, $f(2) = 1.2484$.

Hence, the smallest root in magnitude lies in the interval (- 1, 0).

Using the secant method $x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f_n - f_{n-1}}\right] f_n, \quad n = 1, 2, \ldots\ldots\ldots$, we get for

$x_0 = -1, \quad x_1 = 0, \quad f_0 = f(x_0) = 1.3791, \quad f_1 = f(x_1) = -3$

$$n = 1: \; x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = -0.6851, \; f_2 = f(x_2) = -0.4836$$

$$n = 2: \; x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = -0.8167, \; f_3 = f(x_3) = 0.2468$$

$$n = 3: \; x_4 = x_3 - \left[\frac{x_3 - x_2}{f_3 - f_2}\right] f_3 = -0.7723, \; f_4 = f(x_4) = -0.0081$$

$$n=4:\ x_5 = x_4 - \left[\frac{x_4 - x_3}{f_4 - f_3}\right] f_4 = -0.7737,\ f_5 = f(x_5) = 0.0001$$

$$n=5:\ x_6 = x_5 - \left[\frac{x_5 - x_4}{f_5 - f_4}\right] f_5 = -0.7737$$, which is correct to three decimal places.

(b) We have $x = N^{1/4}$. Take $f(x) = x^4 - N = 0$. Let ξ be the exact root. Therefore, $\xi^4 = N$.

Substituting $x_{n+1} = \xi + \epsilon_{n+1},\ x_n = \xi + \epsilon_n,\ N = \xi^4$ in the given formula, we get

$$\xi + \epsilon_{n+1} = a(\xi + \epsilon_n) + b\xi^4/(\xi + \epsilon_n)^3 + c\xi^8/(\xi + \epsilon_n)^7$$
$$= a(\xi + \epsilon_n) + b\xi[1 + \epsilon_n/\xi]^{-3} + c\xi[1 + \epsilon_n/\xi]^{-7}$$

Expanding in binomial series and simplifying, we get

$$\epsilon_{n+1} = (-1 + a + b + c)\xi + (a - 3b - 7c)\epsilon_n + (6b + 28c)\epsilon_n^2/\xi - (10b + 84c)\epsilon_n^3/\xi^2 + ...$$

For the method to be of highest order, we have

$$a + b + c = 1,\ a - 3b - 7c = 0,\ 6b + 28c = 0$$

Solving these equations, we get $a = 21/32,\ b = 14/32,\ c = -3/32$ and

$$\epsilon_{n+1} = -(10b + 84c)\epsilon_n^3/\xi^2 + O(\epsilon_n^4) = 7\epsilon_n^3/(2\xi^2) + O(\epsilon_n^4)$$

Hence, the method is of third order.

3. (a) We have $f(x, y) = x^2 + 3y^2 + 2xy - 2.51,\quad g(x, y) = 2x^2 + y^2 - 5xy - 12.83$.

We obtain the Jacobian matrix

$$\mathbf{J}_n = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}_n = \begin{bmatrix} 2x_n + 2y_n & 6y_n + 2x_n \\ 4x_n - 5y_n & 2y_n - 5x_n \end{bmatrix}$$ and

$$\mathbf{J}_n^{-1} = \frac{1}{D}\begin{bmatrix} 2y_n - 5x_n & -(6y_n + 2x_n) \\ -(4x_n - 5y_n) & 2x_n + 2y_n \end{bmatrix}$$

$$D = (2x_n + 2y_n)(2y_n - 5x_n) - (6y_n + 2x_n)(4x_n - 5y_n)$$

Using Newton's method $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \mathbf{J}_n^{-1}\begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}, n = 0,1,.......,$ we get

$$n=0:\quad x_0 = 1.5,\quad y_0 = -1,\quad f_0 = -0.26,\quad g_0 = 0.17,\quad D = 23.5$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.5 \\ -1 \end{pmatrix} - \frac{1}{23.5}\begin{pmatrix} -9.5 & 3 \\ -11 & 1 \end{pmatrix}\begin{pmatrix} -0.26 \\ 0.17 \end{pmatrix} = \begin{pmatrix} 1.3732 \\ -1.1289 \end{pmatrix}$$

(b) We write the given coefficient matrix **A** as

$\mathbf{A} = \mathbf{L}\mathbf{L}^T$ where $\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$. We obtain

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 3 \\ 1 & 3 & 37/2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing element by element, we get,

First row: $l_{11}^2 = 4 \Rightarrow l_{11} = 2$,

$l_{11}l_{21} = 0 \Rightarrow l_{21} = 0, \quad l_{11}l_{31} = 1 \Rightarrow l_{31} = 1/2$

Second row: $l_{21}^2 + l_{22}^2 = 4 \Rightarrow l_{22}^2 = 4 \Rightarrow l_{22} = 2$,

$l_{21}l_{31} + l_{22}l_{32} = 3 \Rightarrow l_{32} = 3/2$

Third row: $l_{31}^2 + l_{32}^2 + l_{33}^2 = 37/2 \Rightarrow l_{33}^2 = 16 \Rightarrow l_{33} = 4$

Hence, we obtain $\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1/2 & 3/2 & 4 \end{bmatrix}$

Write the given system of equations $\mathbf{A\,x = b}$ as $\mathbf{L\,L}^T\mathbf{x = b}$

or $\mathbf{L}^T\mathbf{x = z}$, $\mathbf{L\,z = b}$

From $\mathbf{L\,z = b}$, that is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1/2 & 3/2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 59 \end{bmatrix}$

we obtain using forward substitution $z_1 = 5/2, \quad z_2 = 13/2, \quad z_3 = 12$

From $\mathbf{L}^T\mathbf{x = z}$, that is $\begin{bmatrix} 2 & 0 & 1/2 \\ 0 & 2 & 3/2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 13/2 \\ 12 \end{bmatrix}$

we obtain using back substitution $x_3 = 3, \quad x_2 = 1, \quad x_1 = 1/2$

4. (a) We obtain from the augmented matrix

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & -3 & 2 & 3 \\ 2 & 6 & 8 & -1 \\ 4 & -3 & 1 & 4.25 \end{array}\right] R_1 \approx R_3 \approx \left[\begin{array}{ccc|c} 4 & -3 & 1 & 4.25 \\ 2 & 6 & 8 & -1 \\ 1 & -3 & 2 & 3 \end{array}\right] \begin{matrix} \\ R_2 - R_1/2 \\ R_3 - R_1/4 \end{matrix}$$

$$\approx \left[\begin{array}{ccc|c} 4 & -3 & 1 & 17/4 \\ 0 & 15/2 & 15/2 & -25/8 \\ 0 & -9/4 & 7/4 & 31/16 \end{array}\right] \quad R_3 + 3R_2/10$$

$$\approx \left[\begin{array}{ccc|c} 4 & -3 & 1 & 17/4 \\ 0 & 15/2 & 15/2 & -25/8 \\ 0 & 0 & 4 & 1 \end{array}\right]$$

Using back substitution, we get $x_3 = 1/4, \quad x_2 = -2/3, \quad x_1 = 1/2$

(b) We obtain from the augmented matrix

$$[\mathbf{A} \,|\, \mathbf{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1/2 \\ R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3/2 & -2 & -1/2 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_1 - 2R_2/3 \\ R_3 - 2R_2 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 2 & 0 & 10/3 & 4/3 & -2/3 & 0 \\ 0 & 3/2 & -2 & -1/2 & 1 & 0 \\ 0 & 0 & 5 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ R_1 - 2R_3/3 \\ R_2 + 2R_3/5 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 4/3 & 2/3 & -2/3 \\ 0 & 3/2 & 0 & -1/2 & 1/5 & 2/5 \\ 0 & 0 & 5 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \\ R_1/2,\ 2R_2/3,\ R_3/5 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & 2/15 & 4/15 \\ 0 & 0 & 1 & 0 & -2/5 & 1/5 \end{array}\right]$$

Hence, $$\mathbf{A}^{-1} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ -1/3 & 2/15 & 4/15 \\ 0 & -2/5 & 1/5 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 10 & 5 & -5 \\ -5 & 2 & 4 \\ 0 & -6 & 3 \end{bmatrix}$$

5. (a) From the given system of equations, we obtain

$$x_1^{(k+1)} = \left[4 + 2x_2^{(k)} - x_3^{(k)}\right]/4, \quad x_2^{(k+1)} = \left[0.75 - x_1^{(k+1)} - x_3^{(k)}\right]/2$$

$$x_3^{(k+1)} = \left[5.5 - 3x_1^{(k+1)} + 3x_2^{(k+1)}\right]/5, \quad k = 0,1,..........$$

We have $x_1^{(0)} = 0.6, \quad x_2^{(0)} = -0.2, \quad x_3^{(0)} = 0.5$. From the above equations, we obtain

$$k = 0: \quad x_1^{(1)} = (4 - 0.4 - 0.5)/4 = 0.775, \quad x_2^{(1)} = (0.75 - 0.775 - 0.5)/2 = -0.2625,$$

$$x_3^{(1)} = [5.5 - 3(0.775) + 3(-0.2625)]/5 = 0.4775$$

$$k=1: \quad x_1^{(2)} = [4+2(-0.2625)-(0.4775)]/4 = 0.749375,$$
$$x_2^{(2)} = [0.75-(0.749375)-0.4775]/2 = -0.238438,$$
$$x_3^{(2)} = [5.5-3(0.749375)+3(-0.238438)]/5 = 0.507313$$
$$k=2: \quad x_1^{(3)} = [4+2(-0.238438)-(0.507312)]/4 = 0.753953,$$
$$x_2^{(3)} = [0.75-(0.753953)-0.507312]/2 = -0.255633,$$
$$x_3^{(3)} = [5.5-3(0.753953)+3(-0.255632)]/5 = 0.494248$$
$$k=3: \quad x_1^{(4)} = [4+2(-0.255632)-(0.494249)]/4 = 0.748621,$$
$$x_2^{(4)} = [0.75-(0.748622)-0.494249]/2 = -0.246435,$$
$$x_3^{(4)} = [5.5-3(0.748622)+3(-0.246436)]/5 = 0.502966$$

After four iterations, we obtain the solution

$$x_1 = 0.748621, \quad x_2 = -0.246435, \quad x_3 = 0.502966$$

The iteration matrix associated with the Gauss-Seidal method is given by $\mathbf{H} = -(\mathbf{D}+\mathbf{L})^{-1}\,\mathbf{U}$. We have

$$\mathbf{D}+\mathbf{L} = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & -3 & 5 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\mathbf{D}+\mathbf{L})^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ -1/8 & 1/2 & 0 \\ -9/40 & 3/10 & 1/5 \end{bmatrix}$$. We obtain

$$\mathbf{H} = -\begin{bmatrix} 1/4 & 0 & 0 \\ -1/8 & 1/2 & 0 \\ -9/40 & 3/10 & 1/5 \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & -1/4 \\ 0 & -1/4 & -3/8 \\ 0 & -9/20 & -3/40 \end{bmatrix}$$

Characteristics equation of the matrix **H** is given by

$$|\mathbf{H}-\lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1/2 & -1/4 \\ 0 & -(1/4)-\lambda & -3/8 \\ 0 & -9/20 & -(3/40)-\lambda \end{vmatrix} = 0 = \lambda(40\lambda^2+13\lambda-6)$$

The eigen values of **H** are 0, 0.2575, -0.5825

Since spectral radius is $\rho(\mathbf{H}) = 0.5825 < 1$, the method converges.

Rate of convergence = $v = -\log_{10}(\rho(\mathbf{H})) = 0.2347$

(Note that rate of convergence can also be written as $v = -\ln(\rho(\mathbf{H})) = 0.5404$)

(b) The largest off diagonal element in magnitude in **A** is $a_{13} = 2$. We obtain

$$\tan 2\theta = 2a_{13}/(a_{11}-a_{33}) = \infty \Rightarrow \theta = \pi/4$$

$$\text{Define } \mathbf{S}_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \text{ and}$$

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A} \mathbf{S}_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & \sqrt{2} & 2 \\ \sqrt{2} & 4 & \sqrt{2} \\ 2 & \sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & \sqrt{2} & 0 \\ 2 & 4 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the largest off-diagonal element in magnitude in $\mathbf{A}_1$ is $a_{12} = 2$. We obtain

$$\tan 2\theta = 2a_{12}/(a_{11} - a_{22}) = \infty \Rightarrow \theta = \pi/4$$

$$\text{Define } \mathbf{S}_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$\mathbf{A}_2 = \mathbf{S}_2^T \mathbf{A}_1 \mathbf{S}_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & -\sqrt{2} & 0 \\ 3\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the eigen values of **A** are 6, 2 and 0.

6. (a) We have

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \frac{1}{40} \begin{pmatrix} 11 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 11 \end{pmatrix}$$

and $\mathbf{V}_0 = [0.4, -0.9, 0.4]^T$

We have $\mathbf{Y}_{k+1} = \mathbf{A}^{-1}\mathbf{V}_k$, and

$\mathbf{V}_{k+1} = \mathbf{Y}_{k+1}/m_{k+1}$, ($m_{k+1}$ is largest element in magnitude in $\mathbf{Y}_{k+1}$)

We obtain

$$k = 0: \ \mathbf{Y}_1 = \frac{1}{40}\begin{pmatrix} 11 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 11 \end{pmatrix}\begin{pmatrix} 0.4 \\ -0.9 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.21 \\ -0.44 \\ 0.21 \end{pmatrix}$$

$m_1 = 0.44$ and $\mathbf{V}_1 = \mathbf{Y}_1/m_1 = [0.4773, -1, 0.4773]^T$

$$k = 1: \ \mathbf{Y}_2 = \frac{1}{40}\begin{pmatrix} 11 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 11 \end{pmatrix}\begin{pmatrix} 0.4773 \\ -1 \\ 0.4773 \end{pmatrix} = \begin{pmatrix} 0.2432 \\ -0.4955 \\ 0.2432 \end{pmatrix}$$

$m_2 = 0.4955$ and $\mathbf{V}_2 = \mathbf{Y}_2/m_2 = [0.4908, -1, 0.4908]^T$

$$k = 2: \ \mathbf{Y}_3 = \frac{1}{40}\begin{pmatrix} 11 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 11 \end{pmatrix}\begin{pmatrix} 0.4908 \\ -1 \\ 0.4908 \end{pmatrix} = \begin{pmatrix} 0.2472 \\ -0.4982 \\ 0.2472 \end{pmatrix}$$

After three iterations, we obtain the ratios as

$$\mu = (\mathbf{Y}_3)_r / (\mathbf{V}_2)_r = [0.5037, \ 0.4982, \ 0.5037]$$

Therefore, $\mu \approx 0.5$ and $\lambda = 1/\mu = 2$

Hence, the smallest eigen value in magnitude of $\mathbf{A}$ is 2.

(b) From the given matrix, we obtain $\tan 2\theta = a_{13} / a_{12} = -1 \Rightarrow \theta = -\pi/4$

Define $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A} \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 4 & -4 \\ 4 & 1 & 2 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 8/\sqrt{2} & 0 \\ 4 & -1/\sqrt{2} & 3/\sqrt{2} \\ -4 & 1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4\sqrt{2} & 0 \\ 4\sqrt{2} & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & 0 \\ c_1 & b_2 & c_2 \\ 0 & c_2 & b_3 \end{bmatrix}$$

This is the required tri-diagonal form. We obtain the sturm sequence as

$f_0 = 1, \quad f_1 = \lambda - b_1 = \lambda - 1$

$f_2 = (\lambda - b_2) f_1 - c_1^2 f_0 = (\lambda + 1)(\lambda - 1) - 32 = \lambda^2 - 33,$

$f_3 = (\lambda - b_3) f_2 - c_2^2 f_1 = (\lambda - 3)(\lambda^2 - 33)$

Eigen values are the roots of $f_3 = 0$. Hence, the eigen values are $\lambda = 3, \pm\sqrt{33}$.

Therefore, the smallest eigen value in magnitude is $\lambda = 3$.

7. (a) The maximum error in linear interpolation is given by $h^2 M_2 / 8$, where

$$M_2 = \max_I |f''(x)| = \max_{1 \le x \le 2} \left|20(1+x)^3\right| = 540$$

We choose h such that $h^2 (540/8) < 5 \times 10^{-4}$. We get $h < 0.00272$

Hence, the largest step size that can be used is $h \approx 0.0027$.

(b) Let the polynomial be $f(x) = a_0 x^3 + a_1 x^2 + a_2 x + a_3$

From $f(0) = 1$, we get $a_3 = 1$; $f(1) = 5$, we get $a_0 + a_1 + a_2 + a_3 = 5$

$f'(0) = 2$, we get $a_2 = 2$; $f'(1) = 4$, we get $3a_0 + 2a_1 + a_2 = 4$

Solving the above equations, we get $a_0 = -2, a_1 = 4, a_2 = 2, a_3 = 1$
and the polynomial is

$$f(x) = -2x^3 + 4x^2 + 2x + 1$$

8. (a) We have $f[x_0, x_1] = \dfrac{1}{x_1 - x_0}\left[f(x_1) - f(x_0)\right] = \dfrac{1}{x_1 - x_0}\left[u(x_1)v(x_1) - u(x_0)v(x_0)\right]$

$$= \frac{1}{x_1 - x_0}\left[v(x_1)\{u(x_1) - u(x_0)\} + u(x_0)\{v(x_1) - v(x_0)\}\right]$$

$$= v(x_1)\left[\frac{u(x_1) - u(x_0)}{x_1 - x_0}\right] + u(x_0)\left[\frac{v(x_1) - v(x_0)}{x_1 - x_0}\right]$$

$$= v(x_1)u[x_0, x_1] + u(x_0)v[x_0, x_1].$$

(b) We have $\Delta = E - 1$ and $\nabla = 1 - E^{-1}$. We get

L.H.S. $= \Delta + \nabla = E - 1 + 1 - E^{-1} = E - E^{-1}$

R.H.S. $= \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta} = \dfrac{E-1}{1-E^{-1}} - \dfrac{1-E^{-1}}{E-1} = \dfrac{E(1-E^{-1})}{1-E^{-1}} - \dfrac{E^{-1}(E-1)}{E-1} = E - E^{-1} =$ L.H.S.

(c) Since the points are equispaced with $h = 0.1$, we can use Newton difference interpolation. We first construct the forward difference table. We have

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.1	0.93			
0.2	0.92	- 0.01		
0.3	0.97	0.05	0.06	
0.4	1.08	0.11	0.06	0
0.5	1.25	0.17	0.06	0
0.6	1.48	0.23	0.06	0

using the Newton forward difference interpolation and the given data, we obtain

$$f(x) = 0.93 + \frac{x-0.1}{0.1}(-0.01) + \frac{(x-0.1)(x-0.2)}{2(0.01)}(0.06) = 3x^2 - x + 1$$

9. (a) We need an approximation of the form $y + a + (b/x)$. We determine a and b such that

$I(a,b) = \sum_{i=1}^{5}[y_i - \{a + (b/x_i)\}]^2 =$ minimum. We obtain the normal equations

$$\frac{\partial I}{\partial a} = -2\sum\left[y_i - a - \frac{b}{x_i}\right] = 0 \quad \text{or} \quad \sum y_i - 5a - b\sum\frac{1}{x_i} = 0$$

$$\frac{\partial I}{\partial b} = -2\sum\left[y_i - a - \frac{b}{x_i}\right]\frac{1}{x_i} = 0 \quad \text{or} \quad \sum\frac{y_i}{x_i} - a\sum\frac{1}{x_i} - b\sum\frac{1}{x_i^2} = 0$$

From the given data, we obtain

$\sum y_i = 16.7,\ \sum(1/x_i) = 2.2833,\ \sum(1/x_i^2) = 1.4636,\ \sum(y_i/x_i) = 8.925$

Hence, we have the normal equations

$$5a + 2.2833b = 16.7, \quad 2.2833a + 1.4636b = 8.925$$

Solving these equations, we obtain $a = 1.9305$ and $b = 3.0863$

(b) Using the given formula and the given data, we get for $x = 0.4$ and

$h = 0.2: \quad y''(0.4) = [y(0.2) - 2y(0.4) + y(0.6)]/(0.2)^2 = -4.0825$

$h = 0.1: \quad y''(0.4) = [y(0.3) - 2y(0.4) + y(0.5)]/(0.1)^2 = -4.1600$

Using the Richardson extrapolation scheme, we obtain the improved value of $y''(h)$ as $[4y''(h/2) - y''(h)]/3$. We get

$$y''(0.4) = [4(-4.1600) - (-4.0825)]/3 = -4.1858$$

10. (a) The method is exact for $f(x) = 1$ and $f(x) = x$. Making the method exact for $f(x) = x^2$, we get $\int_{x_0}^{x_1} x^2 dx = (x_1^3 - x_0^3)/3 = h[x_0^2 + x_1^2]/2 + ph^2[2x_0 - 2x_1]$

Writing $x_1 = x_0 + h$, we obtain

$\frac{1}{3}(3hx_0^2 + 3h^2x_0 + h^3) = \frac{h}{2}(2x_0^2 + 2hx_1 + h^2) - 2ph^3$, which gives $p = 1/12$.

We write the integral $\int_a^b f(x)\,dx$ as

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + ... + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Replacing each integral on the right side by the given formula, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \left[\frac{h}{2}(f_0+f_1)+\frac{h^2}{12}(f_0'-f_1')\right]+\left[\frac{h}{2}(f_1+f_2)+\frac{h^2}{12}(f_1'-f_2')\right]$$

$$+\,.........+\left[\frac{h}{2}(f_{n-1}+f_n)+\frac{h^2}{12}(f_{n-1}'-f_n')\right]$$

$$=\frac{h}{2}\left[f_0+2(f_1+f_2+..........f_{n-1})+f_n\right]+\frac{h^2}{12}(f_0'-f_n')$$

which is the required composite rule.

(b) Using $x=(t+5)/2$ we change the limits of integration from [2, 3] to [-1, 1]. Hence,

we get $I=\int_{-1}^{1} f(t)dt,$ where $f(t)=\frac{1}{2}\left[\frac{\cos 2(t+5)}{1+\sin((t+5)/2)}\right]$

Using Gauss two-point formula $\int_{-1}^{1} f(x)dx = f(1/\sqrt{3})+f(-1/\sqrt{3})$

we get $I=[0.117755-0.464401]/2=-0.173323$

Using Gauss three-point formula $\int_{-1}^{1} f(x)dx = \left[5f\left(-\sqrt{3/5}\right)+8f(0)+5f\left(\sqrt{3/5}\right)\right]/9$

we get $I=[5(-0.302693)+8(-0.524921)+5(0.420090)]/18=-0.200688$

11. (a) Taylor series second order method is given by

$$y_{n+1}=y_n+hy_n'+h^2y_n''/2$$
$$y_{n+1}'=y_n'+hy_n''+h^2y_n'''/2,\quad n=0,1,..........$$

We have $y''=\cos x-3y'-2y,\quad y'''=-\sin x-3y''-2y'$ and $h=0.2$. We get

$n=0:\quad x_0=0,\ y_0=1,\ \ y_0'=1,\ \ y_0''=-4,\ \ y_0'''=10$ and

$$y_1\approx y(0.2)=y_0+(0.2)y_0'+(0.2)^2y_0''/2=1.12$$
$$y_1'\approx y'(0.2)=y_0'+(0.2)y_0''+(0.2)^2y_0'''/2=0.4$$

$n=1:\quad x_1=0.2,\ y_1=1.12,\ \ y_1'=0.4,\ \ y_1''=-2.4599,\ \ y_1'''=6.3811$ and

$$y_2\approx y(0.4)=y_1+(0.2)y_1'+(0.2)^2y_1''/2=1.1508$$
$$y_2'\approx y'(0.4)=y_1'+(0.2)y_1''+(0.2)^2y_1'''/2=0.0356$$

(b) We have $x_0=1$, $y_0=2$, $h=0.2$ and $f(x, y)=(y+2x)/(y+3x)$.
Using the given method, we get for

$n=0:\quad k_1=hf(x_0,\ y_0)=0.2f(1,\ 2)=0.16$

$$k_2=hf(x_0+h/2,\ y_0+k_1/2)=0.2f(1.1,\ 2.08)=0.1591$$
$$k_3=hf(x_0+h,\ y_0-k_1+2k_2)=0.2f(1.2,\ 2.1582)=0.1583$$
$$y_1\approx y(1.2)=y_0+(k_1+4k_2+k_3)/6=2.1591$$

Subject **NUMERICAL COMPUTING**

Code **C-09 / T-09 (December 2003)**

1. (a) Absolute error = $|\text{true value - approximate value}|$

$$= |3.1415926 - 3.1428571| = 0.0012645$$

Relative error = $\dfrac{\text{Absolute error}}{|\text{true value}|} = 0.000402$ **Answer: A**

(b) The rate of convergence of Newton-Raphson method is 2. Hence $s = 2$

Answer: A

(c) Using Gerschgorin theorem, we find that

$$|\lambda| \le \max\left[\frac{7}{12}, \frac{5}{6}, \frac{3}{4}\right] = \frac{5}{6} < 1, \text{ Hence, } \rho(\mathbf{A}) < 1$$ **Answer: D**

(d) The maximum error in linear interpolation is $(h^2/8)M_2$

where $M_2 = \max_I |f''(x)| = \max_{0 \le x \le \pi/4} |-\sin x| = 1/\sqrt{2}$.

We choose h such that

$$\frac{h^2}{8}\left(\frac{1}{\sqrt{2}}\right) < 0.00000005 \quad \text{or} \quad h < 0.00075$$

Hence, largest value of h is 0.00075 **Answer: D**

(e) Jacobian matrix $= \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 2x-1 & 2y-1 \\ 2x & -2y-1 \end{bmatrix}$

At the point (1, 1), we get, Jacobian matrix $= \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ **Answer: B**

(f) Since f is a polynomial of degree k, all the divided differences of order k are equal and divided differences of order greater than k are zeros. **Answer: C**

(g) Trapezoidal rule is exact for polynomials of degree upto one. **Answer: C**

(h) Mid-point rule is given by : $y_{n+1} = y_{n-1} + 2hf(x_n, y_n)$

For $n = 1$, we get $y_2 = y_0 + 2h\ f(x_1, y_1)$

We calculate $y_1 = y(0.2)$ from the exact solution $y(x) = 1/(1+x^2)$

We obtain $y_1 = 1/(1+(0.2)^2) = 0.9615$

Hence, from the given formula, we get for $h = 0.2$, $y_0 = 1$ and $x_1 = 0.2$

$$y_2 \approx y(0.4) = y_0 + 2h(-2x_1y_1^2) = 0.8520$$ **Answer: B**

2 (a) Substituting $x_n = A\xi^n$, we obtain the characteristic equation

$$\xi^2 + 0.2\xi - 0.99 = 0 \quad \text{or} \quad 100\xi^2 + 20\xi - 99 = 0$$

The roots of the characteristic equation are -1.1 and 0.9

The general solution is written as $x_n = c_1(-1.1)^n + c_2(0.9)^n$

Using the given conditions, we obtain

$$n = 0: \quad c_1 + c_2 = 1, \qquad n = 1: \quad -1.1c_1 + 0.9c_2 = 0.9$$

Solving, we get $c_1 = 0$ and $c_2 = 1$ and $x_n = (0.9)^n$

Since one root of the characteristic equation is greater than 1 in magnitude, computation will not be stable.

(b) We have $f(x) = 3x - \cos x - 1$, we find the $f(x) < 0$ for all $x < 0$. We obtain

$$f(0) = -2 < 0 \text{ and } f(1) = 3 - \cos 1 - 1 = 1.46 > 0.$$

Therefore, the smallest root in magnitude lies in (0, 1).

Using the Newton-Raphson method, we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}, n = 0,1,..........$$

$n = 0$: $x_0 = 0.5$, $f(x_0) = -0.377583$, $f'(x_0) = 3.479426$ and $x_1 = 0.608519$

$n = 1$: $x_1 = 0.608519$, $f(x_1) = 0.005060$, $f'(x_1) = 3.571653$ and $x_2 = 0.607102$

3. (a) **Without pivoting**

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cc|c} \in & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \approx \left[\begin{array}{cc|c} \in & 1 & 1 \\ 0 & 1\text{-}1/\in & 2\text{-}1/\in \end{array}\right]$$

Using back substitution, we get

$$x_2 = \frac{2 - 1/\in}{1 - 1/\in} = \frac{2\in - 1}{\in - 1}, \quad x_1 = \frac{1}{\in}[1 - x_2] = -\frac{1}{\in - 1}$$

With pivoting

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cc|c} \in & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 1 & 2 \\ \in & 1 & 1 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1\text{-}\in & 1\text{-}2\in \end{array}\right]$$

Using back substitution, we get

$$x_2 = \frac{1 - 2\in}{1 - \in}, x_1 = 2 - x_2 = \frac{1}{1 - \in} = -\frac{1}{\in - 1}$$

Results in both cases are same, since we are using exact arithmetic.

(b) The iteration matrix in the Jacobi method is given by $\mathbf{M} = -\mathbf{D}^{-1}(\mathbf{L}+\mathbf{U})$. We obtain from the given matrix $\mathbf{A}$, $\mathbf{M} = -\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}0 & k\\2k & 0\end{bmatrix} = \begin{bmatrix}0 & -k\\-2k & 0\end{bmatrix}$.

The eigen values of $\mathbf{M}$ are obtained from

$$|\mathbf{M}-\lambda\mathbf{I}| = \begin{vmatrix}-\lambda & -k\\-2k & -\lambda\end{vmatrix} = \lambda^2 - 2k^2 = 0 \quad \text{or} \quad \lambda = \pm\sqrt{2}k.$$

For convergence of Jacobi method, we have

$$\rho(\mathbf{M}) = \sqrt{2}\,|k| < 1. \quad \text{Hence, } |k| < 1/\sqrt{2}$$

4. (a) We obtain from the augmented matrix

$$(\mathbf{A}|\mathbf{b}) = \left[\begin{array}{ccc|c}4 & 1 & 1 & 5\\2 & 5 & -2 & 9\\2 & 3 & 6 & 13\end{array}\right] \begin{array}{l} \\ \\ R_2 - R_1/2\\ R_3 - R_1/2\end{array}$$

$$\approx \left[\begin{array}{ccc|c}4 & 1 & 1 & 5\\0 & 9/2 & -5/2 & 13/2\\0 & 5/2 & 11/2 & 21/2\end{array}\right] \; R_3 - 5R_2/9$$

$$\approx \left[\begin{array}{ccc|c}4 & 1 & 1 & 5\\0 & 9/2 & -5/2 & 13/2\\0 & 0 & 62/9 & 62/9\end{array}\right]$$

Using back substitution, we get

$$z = 1,\; y = \frac{2}{9}\left[\frac{13}{2} + \frac{5}{2}z\right] = 2,\; x = \frac{1}{4}[5 - y - z] = \frac{1}{2}$$

(b) The Gauss-seidel iteration method in matrix form is given by

$$\mathbf{x}_{n+1} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U}\,\mathbf{x}_n + (\mathbf{D}+\mathbf{L})^{-1}\mathbf{b} \;=\; \mathbf{H}\,\mathbf{x}_n + \mathbf{c}$$

From the given system of equations, we have

$$\mathbf{D}+\mathbf{L} = \begin{bmatrix}4 & 0 & 0\\0 & 5 & 0\\5 & 4 & 10\end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix}0 & 0 & 2\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

We obtain

$$\mathbf{H} = -(\mathbf{D}+\mathbf{L})^{-1}\mathbf{U} = -\frac{1}{200}\begin{pmatrix}50 & 0 & 0\\0 & 40 & 0\\-25 & -16 & 20\end{pmatrix}\begin{pmatrix}0 & 0 & 2\\0 & 0 & 2\\0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & -1/2\\0 & 0 & -2/5\\0 & 0 & 41/100\end{pmatrix}$$

Now eigen values of $\mathbf{H}$ are $\lambda = 0, 0, 0.41$

Since $\rho(\mathbf{H}) = 0.41 < 1$, the method converges. The rate of convergence is given by

$$v = -\log_{10}(\rho(\mathbf{H})) = 0.3872 \quad \text{or} \quad v = -\ell n(\rho(\mathbf{H})) = 0.8916$$

Q5. (a) We have $f(x_1, x_2) = 4x_1^2 - x_2^2, \quad g(x_1, x_2) = 4x_1 x_2^2 - x_1 - 1$

$$\mathbf{J} = \text{Jacobian matrix} = \begin{bmatrix} \partial f / \partial x_1 & \partial f / \partial x_2 \\ \partial g / \partial x_1 & \partial g / \partial x_2 \end{bmatrix} = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1 x_2 \end{bmatrix}$$

$$\mathbf{J}^{-1} = \frac{1}{D}\begin{bmatrix} 8x_1 x_2 & 2x_2 \\ 1 - 4x_2^2 & 8x_1 \end{bmatrix}, \quad D = 64x_1^2 x_2 + 2x_2(4x_2^2 - 1)$$

Using Newton's method : $\begin{bmatrix} x_1^{(n+1)} \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(n)} \\ x_2^{(n)} \end{bmatrix} - \mathbf{J}_n^{-1}\begin{bmatrix} f(x_1^{(n)}, x_2^{(n)}) \\ g(x_1^{(n)}, x_2^{(n)}) \end{bmatrix}, \; n = 0,1,\ldots\ldots$

we obtain

$n = 0: \; x_1^{(0)} = 0, \; x_2^{(0)} = 1, \; f_0 = -1, \; g_0 = -1, \; D = 6$

$$\begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix}$$

$n = 1: \; x_1^{(1)} = 1/3, \; x_2^{(1)} = 1/2, \; f_1 = 7/36, \; g_1 = -1, \; D = 32/9$

$$\begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/2 \end{pmatrix} - \frac{9}{32}\begin{pmatrix} 4/3 & 1 \\ 0 & 8/3 \end{pmatrix}\begin{pmatrix} 7/36 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.541667 \\ 1.25 \end{pmatrix}$$

(b) We write $\mathbf{A} = \mathbf{L}\,\mathbf{L}^T$, where $\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

We obtain

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}\ell_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}\ell_{31} + l_{22}\ell_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing element by element, we obtain

first row: $l_{11}^2 = 2 \Rightarrow l_{11} = \sqrt{2}; \; l_{11}l_{21} = 3 \Rightarrow l_{21} = 3\sqrt{2}/2,$

$l_{11}l_{31} = -1 \Rightarrow l_{31} = -\sqrt{2}/2$

second row: $l_{21}^2 + l_{22}^2 = 1 \Rightarrow l_{22}^2 = -7/2$ which is not possible.

Hence, we cannot use the Choleski method.

Note: For the use of Choleski method, the given coefficient matrix must be positive definite. The given matrix $\mathbf{A}$ is not positive definite matrix since first leading minor = 2 > 0 and second leading minor = $\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7 < 0$

Q6. (a) The largest off-diagonal element in magnitude in $\mathbf{A}$ is $a_{13} = 2$. We obtain

$$\tan 2\theta = \frac{2a_{13}}{a_{11} - a_{33}} = \infty, \quad \text{or} \quad \theta = \frac{\pi}{4}$$

Now define $\mathbf{S}_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

and

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A} \, \mathbf{S}_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3/\sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 2 & 3 & 0 \\ 3/\sqrt{2} & \sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Now, largest off diagonal element in $\mathbf{A}_1$ is $a_{12} = 2$. We obtain

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} = \infty, \quad \text{or} \quad \theta = \frac{\pi}{4}$$

Now define $\mathbf{S}_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and

$$\mathbf{A}_2 = \mathbf{S}_2^T \mathbf{A}_1 \mathbf{S}_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} & -1/\sqrt{2} & 0 \\ 5/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is the diagonal matrix. The eigen values of $\mathbf{A}$ are 5, 1, -1.

The eigen vectors are obtained from

$$\mathbf{S} = \mathbf{S}_1\mathbf{S}_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

Hence, eigen vectors are

$\lfloor 1/2,\ 1/\sqrt{2},\ 1/2 \rfloor^T, \lfloor -1/2,\ 1/\sqrt{2},\ -1/2 \rfloor^T, \lfloor -1/\sqrt{2},\ 0,\ 1/\sqrt{2} \rfloor^T$ respectively.

(corresponding to $\lambda = 5, \lambda = 1, \lambda = -1$)

.

6. (b) We obtain using Given's method and the given matrix $\mathbf{A}$

$$\tan\theta = \frac{a_{13}}{a_{12}} = 1, \text{ or } \theta = \frac{\pi}{4}$$

Define $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

and

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A}\, \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 4/\sqrt{2} & 0 \\ 2 & 3/\sqrt{2} & 1/\sqrt{2} \\ 2 & 3/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is the required tridiagonal form

From $\mathbf{A}_1 = \begin{bmatrix} 1 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & 0 \\ c_1 & b_2 & c_2 \\ 0 & c_2 & b_3 \end{bmatrix}$,

we obtain the Strums sequence

$f_0 = 1$

$f_1 = \lambda - b_1 = \lambda - 1$

$f_2 = (\lambda - b_2)f_1 - c_1^2 f_0 = (\lambda - 3)(\lambda - 1) - 8 = \lambda^2 - 4\lambda - 5$

$f_3 = (\lambda - b_3)f_2 - c_2^2 f_1 = (\lambda + 1)(\lambda^2 - 4\lambda - 5) = (\lambda + 1)(\lambda + 1)(\lambda - 5)$

The eigen values of $\mathbf{A}$ are -1, -1 and 5. The largest eigen values in magnitude of $\mathbf{A}$ is 5.

7. (a) We are given that $f(x) = x - 9^{-x}$. We obtain

x	0	0.5	1
$f(x)$	-1	1/6	8/9

We now construct the Newton's forward difference table

x	$f(x)$	Δf	$\Delta^2 f$
0	-1		
0.5	1/6	7/6	
1	8/9	13/18	-4/9

Using Newton's forward difference interpolation

$$f(x) = f(x_0) + \frac{x - x_0}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{2!h^2}\Delta^2 f_0$$

we obtain for h = 1/2

$$f(x) \approx p(x) = -1 + 2x\left(\frac{7}{6}\right) + 2\left(x - \frac{1}{2}\right)(x)\left(\frac{4}{9}\right) = -\frac{8}{9}x^2 + \frac{25}{9}x - 1$$

Setting $p(x) = 0$, we obtain the solutions $x = 2.7098$ and $x = 0.4152$

(b) Let $p(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1) + a\,x(x+1)(x-1)(x-2)$

This polynomial interpolates the first four points in the given table. We determine a so that this polynomial interpolates at the last point also. We get

$p(3) = 2 - 4 + 12 - 48 + 24a = 10$, or $a = 2$

Hence, the required polynomial is

$p(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1) + 2x(x+1)(x-1)(x-2)$

8. (a) We want an approximation of the form $y = a + (b/x)$ where a and b are constants to be determined such that

$$I(a,b) = \sum_{i=1}^{5}(y_i - a - b/x_i)^2 = \text{minimum}$$

We obtain the normal equation

$$\frac{\partial I}{\partial a} = -2\sum\left[y_i - a - \frac{b}{x_i}\right] = 0, \text{ or } \sum y_i - 5a - b\sum\frac{1}{x_i} = 0$$

$$\frac{\partial I}{\partial b} = -2\sum\left[y_i - a - \frac{b}{x_i}\right]\left(\frac{1}{x_i}\right) = 0, \text{ or } \sum\frac{y_i}{x_i} - a\sum\frac{1}{x_i} - b\sum\frac{1}{x_i^2} = 0$$

From the given data, we obtain

$\sum y_i = 65$, $\sum 1/x_i = 2.2833$, $\sum 1/x_i^2 = 1.4636$, $\sum y_i/x_i = 31.5333$

Hence, we obtain the normal equations

$5a + 2.2833b = 65$

$2.2833a + 1.4636b = 31.5333$

Solving these equations, we get $a = 10.9918$ and $b = 4.3972$.

(b) We have

$$TE = y(x_0 + sh) - ay(x_0) - by(x_0 + h) - ch^2 y''(x_0) - dh^2 y''(x_0 + h)$$

$$= y(x_0) + s\,hy'(x_0) + \frac{s^2h^2}{2} y''(x_0) + \frac{s^3h^3}{6} y'''(x_0) + \frac{s^4h^4}{24} y^{iv}(x_0) + ...$$

$$- a\,y(x_0)\ - b\left[y(x_0) + hy'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{6} y'''(x_0) + \frac{h^4}{24} y^{iv}(x_0) + ... \right]$$

$$- ch^2 y''(x_0) - dh^2\left[y''(x_0) + hy'''(x_0) + \frac{h^2}{2} y^{iv}(x_0) + ... \right].$$

$$= (1 - a - b)y(x_0) + (s - b)hy'(x_0) + \left(\frac{s^2}{2} - \frac{b}{2} - c - d \right)h^2 y''(x_0)$$

$$+ \left(\frac{s^3}{6} - \frac{b}{6} - d \right)h^3 y'''(x_0) + \left(\frac{s^4}{24} - \frac{b}{24} - \frac{d}{2} \right)h^4 y^{iv}(x_0) + ...$$

Setting the coefficients of $y^{(r)}(x_0)$, $r = 0, 1, 2, 3$, equal tozero, we obtain

$$1 - a - b = 0, \qquad s - b = 0$$

$$\frac{s^2}{2} - \frac{b}{2} - c - d = 0, \quad \frac{s^3}{6} - \frac{b}{6} - d = 0$$

Solving these equations, we get

$$a = (1 - s), b = s, c = -s(s - 1)(s - 2)/6 \text{ and } d = s(s^2 - 1)/6.$$

9. (a) Let $g(x) = \int_a^b f(x)dx$ be the quantity which is to be obtained and $g(h/2^r)$ denote the approximate value of $g(x)$ obtained by using the given method with step length $h/2^r$, $r = 0, 1, 2,$ ………. Thus we have

$$g(h) = g(x) + c_1h^4 + c_2h^6 + c_3h^8 + ...$$

$$g\left(\frac{h}{2}\right) = g(x) + \frac{c_1h^4}{16} + \frac{c_2h^6}{64} + \frac{c_3h^8}{256} + ...$$

$$g\left(\frac{h}{2^2}\right) = g(x) + \frac{c_1h^4}{256} + \frac{c_2h^6}{4096} + \frac{c_3h^8}{65536} + ...$$

⋮

Eliminating c_1 from the above equations, we get

$$g^{(1)}(h) = \frac{4^2 g(h/2) - g(h)}{4^2 - 1} = g(x) - \frac{1c_2}{20} h^6 - \frac{1}{16} c_3 h^8 - ....$$

$$g^{(1)}(\frac{h}{2}) = \frac{4^2 g(h/2^2) - g(h/2)}{4^2 - 1} = g(x) - \frac{1c_2}{1280} h^6 - \frac{1}{4096} c_3 h^8 - ....$$

⋮

Eliminating c_2 from the above equations, we get

$$g^{(2)}(h)=\frac{4^3 g^{(1)}(h/2)-g^{(1)}(h)}{4^3-1}=g(x)+\frac{1}{1344}c_3h^8+.................$$

$$\vdots$$

Thus successive higher order results can be obtained from the formula

$$g^{(m)}(h)=\frac{4^{m+1} g^{(m-1)}(h/2)-g^{(m-1)}(h)}{4^{m+1}-1}+O\left(h^{2m+4}\right),\quad g^{(0)}(h)=g(h)$$

Now we evaluate the given integral $I=\int_0^1\frac{x\,dx}{1+x+x^2}$

Using Simpson's rule, we get

$h=1/2:\quad x_0=0,\quad x_1=1/2,\quad x_2=1,\quad f_0=0,\quad f_1=2/7,\quad f_2=1/3$

$$I(h)=\frac{h}{3}\left[f_0+4f_1+f_2\right]=\frac{1}{6}\left[0+\frac{8}{7}+\frac{1}{3}\right]=0.246032$$

$h=1/4:\quad x_0=0,\quad x_1=1/4,\ x_2=2/4,\ x_3=3/4,\quad x_4=1,$

$f_0=0,\quad f_1=4/21,\quad f_2=2/7,\ f_3=12/37,\quad f_4=1/3$

$$I(h/2)=\frac{h}{3}\left[f_0+4\left(f_1+f_3\right)+2f_2+f_4\right]$$

$$=\frac{1}{12}\left[0+4\left(\frac{4}{21}+\frac{12}{37}\right)+2\left(\frac{2}{7}\right)+\frac{1}{3}\right]=0.246997$$

Using Romberg integration, we obtain $I=\dfrac{16I(h/2)-I(h)}{15}=0.247061$

9. (b) The method is exact for $f(x)=1$ and x. Making the method exact for $f(x)=x^2$, we get

$$\int_{x_0}^{x_1}x^2dx=\frac{h}{2}\left[x_0^2+x_1^2\right]+ph^2\left(2x_0-2x_1\right)$$

or $\frac{1}{3}\left(x_1^3-x_0^3\right)=\frac{h}{2}\left(x_0^2+x_1^2\right)+2ph^2(x_0-x_1)$

Since $x_1=x_0+h$, we get

$$\frac{1}{3}\left(3hx_0^2+3h^2x_0+h^3\right)=\frac{h}{2}\left(2x_0^2+2hx_0+h^2\right)-2ph^3$$

Simplifying, we obtain $1-4p=2/3$. Therefore, $p=1/12$.

The error of integration is given by :Error = $\frac{c}{3!}f'''(\xi),\ x_0<\xi<x_1$

where, $c=\int_{x_0}^{x_1}x^3dx-\frac{h}{2}\left(x_0^3+x_1^3\right)-\frac{h^2}{12}\left(3x_0^2-3x_1^2\right)=0$

Therefore error is written as :Error = $\frac{c}{4!}f^{iv}(\xi)$

where $c=\int_{x_0}^{x_1}x^4dx-\frac{h}{2}\left(x_0^4+x_1^4\right)-\frac{h^2}{12}\left(4x_0^3-4x_1^3\right)=\frac{h^5}{30}$.

Hence, Error = $\frac{h^5}{720} f^{1v}(\xi)$.

We can now write $I = \int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{N-1}}^{x_N} f(x)dx$

Replacing each integral by the above integration formula, we get

$$I = \left[\frac{h}{2}(f_0 + f_1) + \frac{h^2}{12}(f_0' - f_1')\right] + \left[\frac{h}{2}(f_1 + f_2) + \frac{h^2}{12}(f_1' - f_2')\right] + ...$$

$$+ \left[\frac{h}{2}(f_{N-1} + f_N) + \frac{h^2}{12}(f_{N-1}' - f_N')\right]$$

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + ... + f_{N-1}) + f_N] + \frac{h^2}{12}(f_0' - f_N')$$

which is the required composite rule.

10. (a) $TE = f'(x_0) - \alpha_0 f(x_0) - \alpha_1 f(x_0 + h) - \alpha_2 f(x_0 + 2h)$

Expanding each term in Taylor series about x_0 and collecting terms of various order derivatives we get

$$TE = -(\alpha_0 + \alpha_1 + \alpha_2) f_0 + [1 - h(\alpha_1 + 2\alpha_2)] f_0'$$

$$-\frac{h^2}{2}(\alpha_1 + 4\alpha_2) f_0'' - \frac{h^3}{6}(\alpha_1 + 8\alpha_2) f_0''' ....$$

We choose α_0, α_1, α_2 such that

$$\alpha_0 + \alpha_1 + \alpha_2 = 0$$

$$\alpha_1 + 2\alpha_2 = 1/h$$

$$\alpha_1 + 4\alpha_2 = 0$$

Solving the above system of equations ,we get

$$\alpha_0 = -3/(2h),\ \alpha_1 = 2/h,\ \alpha_2 = -1/(2h)$$

Hence, we obtain the differentiation method

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_1) - f(x_2)]$$

The error term is given by $TE = -\frac{h^3}{6}(\alpha_1 + 8\alpha_2) f'''(\xi) = \frac{h^2}{3} f'''(\xi), x_0 < \xi < x_2$

(b) We write $I = \int_{-1}^{1} \frac{x(1-x^2)\sin x}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$, where $f(x) = x(1-x^2)\sin x$

Using Gauss-Chebyshev two point method

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx=\frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)\right],\quad \text{we get}$$

$$I=\frac{\pi}{2}\left[-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\sin\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\sin\left(\frac{1}{\sqrt{2}}\right)\right]=\frac{\pi}{2}\left[\frac{1}{\sqrt{2}}\sin\left(\frac{1}{\sqrt{2}}\right)\right]=0.7215652$$

Using Gauss-Chebyshev three point formula

$$\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx=\frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right],\ \text{we get}$$

$$I=\frac{\pi}{3}\left[\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\sin\left(-\frac{\sqrt{3}}{2}\right)+0+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{4}\right)\sin\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{\pi}{3}\left[\frac{\sqrt{3}}{4}\sin\left(\frac{\sqrt{3}}{2}\right)\right]=0.345420$$

11. (a) Euler's method is given by : $y_{i+1}=y_i+hf_i=y_i-2hx_iy_i^2,\ i=0,1,2,3,4$

We obtain for $h = 0.2$

$i = 0$: $x_0 = 0$, $y_0 = 1$,

$$y_1\approx y(0.2)=y_0-2(0.2)x_0y_0^2=1$$

$i = 1$: $x_1 = 0.2$, $y_1 = 1$,

$$y_2\approx y(0.4)=y_1-2(0.2)x_1y_1^2=0.92$$

$i = 2$: $x_2 = 0.4$, $y_2 = 0.92$,

$$y_3\approx y(0.6)=y_2-2(0.2)x_2y_2^2=0.784576$$

$i = 3$: $x_3 = 0.6$, $y_3 = 0.784576$,

$$y_4\approx y(0.8)=y_3-2(0.2)x_3y_3^2=0.636842$$

$i = 4$: $x_4 = 0.8$, $y_4 = 0.636842$,

$$y_5\approx y(1.0)=y_4-2(0.2)x_4y_4^2=0.507060$$

(b) Expanding in Taylor series about the point (x_n, y_n), we get

$$k_1=hf(x_n,y_n)=hf_n$$

$$k_2=hf(x_n+c_2h,y_n+a_2k_1)=hf(x_n+c_2h,y_n+a_2hf_n)$$

$$=h\left[f_n+h(c_2f_x+a_2f\ f_y)_n+\frac{h^2}{2}(c_2^2f_{xx}+2c_2a_2f\ f_{xy}+a_2^2f^2f_{yy})_n+\dots\dots\dots\right]$$

Substituting in the given method, we get

$$y_{n+1}=y_n+(w_1+w_2)hf_n+h^2\left[w_2c_2f_x+w_2a_2f\ f_y\right]_n$$

$$+\frac{h^3}{2}w_2\left[c_2^2 f_{xx}+2c_2 a_2 ff_{xy}+a_2^2 f^2 f_{yy}\right]_n+\text{.........} \qquad (1)$$

We also have,

$$y_{n+1}=y_n+hy_n'+\frac{h^2}{2}y_n''+\frac{h^3}{6}y_n'''+\text{............}$$

$$=y_n+hf_n+\frac{h^2}{2}\left[f_x+f\ f_y\right]_n$$

$$+\frac{h^3}{6}\left[\left(f_{xx}+2f\ f_{xy}+f^2 f_{yy}\right)+f_y\left(f_x+f\ f_y\right)\right]_n+\text{............} \qquad (2)$$

Comparing the coefficients of h and h^2 in (1) and (2), we get,

$$w_1+w_2=1,\ w_2c_2=\frac{1}{2},\ w_2a_2=\frac{1}{2}$$

Solving these equations, we get

$$a_2=c_2, w_2=\frac{1}{2c_2}, w_1=\left(\frac{2c_2-1}{2c_2}\right) \text{ and } c_2\neq 0 \text{ is arbitrary.}$$

Hence, we obtain the method

$$y_{n+1}=y_n+\left(1-\frac{1}{2c_2}\right)k_1+\frac{1}{2c_2}k_2$$

$$k_1=hf\left(x_n, y_n\right)$$

$$k_2=hf\left(x_n+c_2h,\ y_n+c_2k_1\right)$$

The truncation error is given by

$$TE=y\left(x_{n+1}\right)-y_{n+1}$$

$$=h^3\left[\left(\frac{1}{6}-\frac{c_2}{4}\right)\left(f_{xx}+2f\ f_{xy}+f^2 f_{yy}\right)+\frac{1}{6}f_y\left(f_x+f\ f_y\right)_n\right]+O(h^4)$$

Hence, the method is of second order for all values of c_2.

Subject: Numerical Computing

Code: C-09/T-09 (June 2004)

1. (a) For one application of Simpson's rule, we require three nodal points. Since we have $2n+1$, (odd) nodal points, the number of sub-intervals n must be even. **Answer: A**

(b) We have $(dy/y) = x\,dx$. Integrating, we get $\ln y = (x^2/2) + c$. Using the given condition, we obtain $c = 0$. Hence, $y = e^{x^2/2} \approx 1 + (x^2/2)$. **Answer: C**

(c) The method produces exact results for polynomials of degree upto 1. The order of convergence is 2. **Answer: D**

(d) $p(x) = (x^4 - 2x^3 + 2x^2 - 2x + 1) = (x^2 - 2x + 1)(x^2 + 1) = 0$. Hence roots are $x = 1, 1$ and $x = \pm i$. These roots form a complex pair.
Answer: C

(e) Using Gerschgorin theorem, we find that

$$|\lambda| \le \max\left[\frac{7}{12}, \frac{5}{6}, \frac{3}{4}\right] = \frac{5}{6}$$

Hence, spectral radius < 1. **Answer: D**

(f) An n-point Gauss-Legendre method is exact for polynomials of degree upto $2n - 1$. Hence, for $n = 4$, the method will produce exact results for polynomials of degree upto 7. **Answer: D**

(g) The error term of the method is given by

$$TE = y(x_n + h) - y(x_n - h) - 2h\,y'(x_n) \quad = h^3 y'''(x_n)/3 + O(h^4)$$

Hence, order of the method is 2. **Answer: A**

(h) We have $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E-1)f(x)$

Hence, $\Delta = E - 1$. The result $\Delta = E + 1$ is wrong. **Answer: B**

2. (a) We obtain from the augmented matrix

$$(\mathbf{A}\,|\,\mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ -3 & -1 & 8 & A \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 3R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -7 & -2 \\ 0 & 2 & 14 & A+3 \end{array}\right] \begin{array}{l} \\ R_3 + 2R_2 \\ \\ \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -7 & -2 \\ 0 & 0 & 0 & A-1 \end{array}\right]$$

For consistency of the system $A = 1$. For other values of A, the system is inconsistent.

(b). We obtain from the augmented matrix

$$(\mathbf{A}\,|\,\mathbf{b}) = \left[\begin{array}{ccc|c} 1 & 1/2 & 1/3 & 1 \\ 1/2 & 1/3 & 1/4 & 0 \\ 1/3 & 1/4 & 1/5 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - R_1/2 \\ R_3 - R_1/3 \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1/2 & 1/3 & 1 \\ 0 & 1/12 & 1/12 & -1/2 \\ 0 & 1/12 & 4/45 & -1/3 \end{array}\right] \begin{array}{l} \\ R_3 - R_2 \\ \\ \end{array}$$

$$\approx \left[\begin{array}{ccc|c} 1 & 1/2 & 1/3 & 1 \\ 0 & 1/12 & 1/12 & -1/2 \\ 0 & 0 & 1/180 & 1/6 \end{array}\right]$$

Using back substitution, we obtain $x_3 = 30$, $x_2 = -36$, $x_1 = 9$.

3. Gauss-Legendre two-point method is written as

$$\int_{-1}^{1} f(x)\,dx = \lambda_0 f(x_0) + \lambda_1 f(x_1)$$

where λ_0, λ_1, x_0, x_1 are to be determined. Making the method exact for $f(x)=1$, x, x^2 and x^3, we get

$$f(x)=1: \quad \int_{-1}^{1} dx = \lambda_0 + \lambda_1, \quad \text{or} \quad \lambda_0 + \lambda_1 = 2 \qquad (1)$$

$$f(x)=x: \quad \int_{-1}^{1} x\,dx = \lambda_0 x_0 + \lambda_1 x_1, \quad \text{or} \quad \lambda_0 x_0 + \lambda_1 x_1 = 0 \qquad (2)$$

$$f(x)=x^2: \quad \int_{-1}^{1} x^2\,dx = \lambda_0 x_0^2 + \lambda_1 x_0^2, \quad \text{or} \quad \lambda_0 x_0^2 + \lambda_1 x_0^2 = 2/3 \qquad (3)$$

$$f(x)=x^3: \quad \int_{-1}^{1} x^3\,dx = \lambda_0 x_0^3 + \lambda_1 x_0^3, \quad \text{or} \quad \lambda_0 x_0^3 + \lambda_1 x_0^3 = 0 \qquad (4)$$

From (2) and (4) we obtain on eliminating λ_0, $\quad \lambda_1 x_0 (x_0^2 - x_1^2) = 0$

Since $\lambda_1 \neq 0$, $x_0 \neq 0$, $x_0 \neq x_1$ (system becomes inconsistent).

we get $x_0 = -x_1$. From (3) we obtain $(\lambda_0 + \lambda_1) x_0^2 = 2/3$ or $x_0^2 = 1/3$

We obtain $x_0 = 1/\sqrt{3}$, $x_1 = -1/\sqrt{3}$, $\lambda_0 = \lambda_1 = 1$.

Hence, the method becomes

$$\int_{-1}^{1} f(x)\,dx = f\left(1/\sqrt{3}\right) + f\left(-1/\sqrt{3}\right)$$

To evaluate the given integral using this method, we first change the limits of integration from [-2, 2] to (-1, 1). Using the substitution $x = 2t$, we obtain

$$I = \int_{-2}^{2} e^{-x/2} dx = 2\int_{-1}^{1} e^{-t} dt = 2\left[e^{1/\sqrt{3}} + e^{-1/\sqrt{3}}\right] = 4.685392$$

4. (a) We have $x_0 = 1,\ x_1 = 3,\ f(x) = x^2 - x - 2,\ f_0 = f(x_0) = -2,\ f_1 = f(x_1) = 4.$

Since $f_0\ f_1 < 0,\ \xi \in (x_0, x_1)$. Using the method of false position, we get

First iteration: $x_2 = \dfrac{x_1 f_0 - x_0 f_1}{f_0 - f_1} = 1.6667, \quad f_2 = f(x_2) = -0.8888$

Since $f_1\ f_2 < 0,\ \xi \in (x_1, x_2)$. we get

Second iteration: $x_3 = \dfrac{x_2 f_1 - x_1 f_2}{f_1 - f_2} = 1.9091,\ f_3 = f(x_3) = -0.2644$

Since $f_1\ f_3 < 0,\ \xi \in (x_1, x_3)$. we get

Third iteration: $x_4 = \dfrac{x_3 f_1 - x_1 f_3}{f_1 - f_3} = 1.9767$

After three iterations, we obtain the root as 1.9767.

(b) Using Lagrange interpolation and the given data, we obtain

$$f(x) \approx p_3(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}(0) + \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)}(1.7183)$$

$$+ \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)}(6.3891) + \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}(19.0855)$$

$$= \frac{1}{2}\left(x^3 - 5x^2 + 6x\right)(1.7183) - \frac{1}{2}\left(x^3 - 4x^2 + 3x\right)(6.3891) + \frac{1}{6}\left(x^3 - 3x^2 + 2x\right)(19.0855)$$

$= 0.8455\, x^3 - 1.0603\, x^2 + 1.9331\, x$

We obtain $f(1.5) \approx 3.3675$.

5 (a) We have $f(x, y) = x + y^2,\ x_0 = 1,\ y_0 = 2,\ h = 0.1$

Using the classical fourth order Runge-Kutta method, we get

$k_1 = hf(x_0, y_0) = 0.1 f(1, 2) = 0.5$

$k_2 = hf(x_0 + h/2,\ y_0 + k_1/2) = 0.1 f(1.05, 2.25) = 0.61125$

$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2) = 0.1 f(1.05, 2.305625) = 0.636591$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = 0.1 f(1.1, 2.636591) = 0.805161$

$y_1 \approx y(1.1) = y_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 2.633474$

Now, $x_1 = 1.1,\ y_1 = 2.633474$. We get

$k_1 = hf(x_1, y_1) = 0.1 f(1.1, 2.633474) = 0.803518$

$$k_2 = hf\left(x_1 + h/2,\ y_1 + k_1/2\right) = 0.1f(1.15, 3.035233) = 1.036264$$
$$k_3 = hf\left(x_1 + h/2,\ y_1 + k_2/2\right) = 0.1f(1.15, 3.151606) = 1.108263$$
$$k_4 = hf\left(x_1 + h,\ y_1 + k_3\right) = 0.1f\left(1.2, 3.741737\right) = 1.520060$$
$$y_2 \approx y(1.2) = y_1 + \left(k_1 + 2k_2 + 2k_3 + k_4\right)/6 = 3.735579$$

(b) We first write the given method in the form

$$x_{k+1} = x_k - \left[\frac{x_k - x_0}{f(x_k) - f(x_0)}\right] f(x_k)$$

Substituting $x_k = \xi + \in_k$ and $x_0 = \xi + \in_0$, we get

$$\in_{k+1} = \in_k - \left[\frac{\in_k - \in_0}{f(\xi+\in_k) - f(\xi+\in_0)}\right] f(\xi+\in_k)$$
$$= \in_k - \frac{[\in_k - \in_0]\left[\in_k f'(\xi) + \in_k^2 f''(\xi)/2 + \ldots\ldots\ldots\ldots\right]}{(\in_k - \in_0) f'(\xi) + \left(\in_k^2 - \in_0^2\right) f''(\xi)/2 + \ldots..}$$

since $f(\xi) = 0$. Cancelling $(\in_k - \in_0)$, we get

$$\in_{k+1} = \in_k - \left[\in_k + C_2 \in_k^2 + \ldots\ldots\ldots\ldots..\right]\left[1 + \{(\in_k + \in_0)C_2 + \ldots\ldots..\}\right]^{-1}$$
$$= \in_k - \left[\in_k + C_2 \in_k^2 + \ldots\ldots\ldots\ldots..\right]\left[1 - (\in_k + \in_0)C_2 + \ldots\ldots...\right]$$

where $C_2 = f''(\xi)/(2f'(\xi))$

Therefore, we get $\in_{k+1} = \in_k - \left[\in_k - C_2 \in_0 \in_k + O\left(\in_k^2 \in_0 + \in_k \in_0^2\right)\right]$

$= C_2 \in_0 \in_k + O\left(\in_k^2 \in_0 + \in_k \in_0^2\right)$

Hence, $\in_{k+1} = c \in_k$, where $c = C_2 \in_0$

Therefore, the method has linear rate of convergence.

6. (a) We have $f(x) = x/\sin x$ and $f(0) = \lim_{x\to 0}(x/\sin x) = 1.$

From the trapezoidal rule, we get

$h = 1/2$: $x_0 = 0$, $x_1 = 1/2$, $f_0 = 1$, $f_1 = 1.042915$ and

$$I = h[f_0 + f_1]/2 = 0.510729$$

$h = 1/4$: $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $f_0 = 1$, $f_1 = 1.010493$, $f_2 = 1.042915$ and

$$I = h[f_0 + 2f_1 + f_2]/2 = 0.507988$$

$h = 1/8$: $x_0 = 0$, $x_1 = 1/8$, $x_2 = 2/8$, $x_3 = 3/8$, $x_4 = 1/2$, $f_0 = 1$, $f_1 = 1.002609$, $f_2 = 1.010493$, $f_3 = 1.023828$, $f_4 = 1.042915$ and

$$I = h[f_0 + 2(f_1 + f_2 + f_3) + f_4]/2 = 0.507298$$

Using Romberg integration

$$I^{(m)}(h) = \frac{4^m I^{(m-1)}(h/2) - I^{(m-1)}(h)}{4^m - 1}, m = 1,2,\ldots\ldots..,\ I^{(0)}(h) = I(h).$$

we obtain the following Romberg table:

h	$0(h^2)$, m=0	$0(h^4)$, m=1	$0(h^6)$, m = 2
1/2	0.510729		
1/4	0.507988	0.507074	
1/8	0.507298	0.507068	0.507068

Hence, $I \approx 0.507068$.

(b) (i) $\Delta\left(\frac{f_i}{g_i}\right) = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i} = \frac{g_i f_{i+1} - f_i g_{i+1}}{g_i g_{i+1}} = \frac{1}{g_i g_{i+1}}\left[g_i\left(f_{i+1} - f_i\right) - f_i\left(g_{i+1} - g_i\right)\right]$

$= \frac{1}{g_i g_{i+1}}\left[g_i \Delta f_i - f_i \Delta g_i\right]$

(ii) $\Delta\left(\frac{1}{f_i}\right) = \frac{1}{f_{i+1}} - \frac{1}{f_i} = \frac{f_i - f_{i+1}}{f_i f_{i+1}}. \quad = -\frac{f_{i+1} - f_i}{f_i f_{i+1}} = -\frac{\Delta f_i}{f_i f_{i+1}}.$

7. (a) If λ is an eigen value of a matrix **A**, then $1/\lambda$ is an eigen value of $\mathbf{A}^{-1}$. Thus the smallest eigen value in magnitude of **A** is the largest eigen value in magnitude of $\mathbf{A}^{-1}$. Thus, we use the power method on $\mathbf{A}^{-1}$ to obtain its largest eigen value μ in magnitude. Then $\lambda = 1/\mu$ is the smallest eigen value in magnitude of **A**.

We take an arbitrary vector $\mathbf{V}_0$ (non-zero) and generate

$\mathbf{Y}_{k+1} = \mathbf{A}^{-1}\, \mathbf{V}_k$

$\mathbf{V}_{k+1} = \mathbf{Y}_{k+1}/\, m_{k+1},$ (where m_{k+1} is the largest element in magnitude in $\mathbf{Y}_{k+1}$)

Then $\mu = \lim_{k\to\infty} \frac{(\mathbf{Y}_{k+1})_r}{(\mathbf{V}_k)_r}$ and $\lambda = 1/\mu$

Since $\mathbf{V}_0 = [0, 0, 0]^T$ is zero vector, we cannot use $\mathbf{V}_0$ to obtain μ.

Hence, in this case solution cannot be obtained. However, if take any other vector $\mathbf{V}_0$ say $\mathbf{V}_0 = [1, 1, 1]^T$, solution can be found. (The question in the present form is wrong).

8. Let **A** be a given real symmetric matrix. The eigen values of **A** are real. There exists a real orthogonal matrix **S** such that $\mathbf{S}^{-1}\, \mathbf{A}\, \mathbf{S}$ is a diagonal matrix **D**. The elements on the diagonal of **D** are the eigen values of **A**. This diagonalization is done by applying a sequence of orthogonal transformations, $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n, \ldots$ as follows:

Among the off diagonal elements, let a_{ik} be the largest element in magnitude.

We define

$$
\mathbf{S} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots\dots & 0 \\ & & \cos\theta & -\sin\theta & & \\ & & & \vdots & & \\ & & \sin\theta & \cos\theta & & \\ 0 & \dots & 0 & & \dots\dots & 1 \end{bmatrix} \text{ ith row}
$$

kth column

Thus **S** is an identity matrix in which the elements in positions (i, i) , (i, k) , (k, i) and (k, k) are written as $\cos\theta$, - $\sin\theta$, $\sin\theta$ and $\cos\theta$ respectively. It can be verified that **S** in an orthogonal matrix $\left(\mathbf{S}_1^{-1} = \mathbf{S}_1^T\right)$.

We consider the 2x2 matrix

$$
\mathbf{S}_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

Now obtain the matrix (since **A** is symmetric, $a_{ik} = a_{ki}$)

$$
\mathbf{A}_1 = \mathbf{S}_1^{-1}\mathbf{A}\mathbf{S}_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a_{ii} & a_{ik} \\ a_{ik} & a_{kk} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}
$$

$$
= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}
$$

where, $p_{11} = a_{ii}\cos^2\theta + a_{kk}\sin^2\theta + 2a_{ik}\sin\theta\cos\theta$

$p_{12} = p_{21} = \left(a_{kk} - a_{ii}\right)\sin\theta\cos\theta + a_{ik}\left(\cos^2\theta - \sin^2\theta\right)$

$p_{22} = a_{ii}\sin^2\theta + a_{kk}\cos^2\theta - 2a_{ik}\sin\theta\cos\theta$

We choose θ such that the matrix $\mathbf{A}_1$ becomes the diagonal matrix.
Setting p_{12}=0, we get

$$
\tan 2\theta = 2a_{ik} / (a_{ii} - a_{kk})
$$

where θ is called the angle of rotation. To obtain the smallest rotation, we take $-\pi/4 \le \theta \le \pi/4$. Now, we find the largest off-diagonal element in $\mathbf{A}_1$ and the procedure is repeated. After r such rotations, we obtain

$$
\mathbf{A}_r = \mathbf{S}_r^{-1}\,\mathbf{S}_{r-1}^{-1}\dots\dots\mathbf{S}_1^{-1}\ \mathbf{A}\,\mathbf{S}_1\mathbf{S}_2\dots\dots\mathbf{S}_r = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}
$$

where, $\mathbf{S} = \mathbf{S}_1\mathbf{S}_2\dots\dots\mathbf{S}_r$.

As $r \to \infty, \mathbf{A}_r$ tends to a diagonal matrix **D** having eigen values on its diagonal.

The columns of **S** give the eigen vectors corresponding to the elements on the diagonal of **D** in that order.

In the given matrix, the largest off-diagonal element in magnitude is either a_{12} or
We take this element as a_{23} (since $a_{22} = a_{33}$ and exact arithmetic can be performed).
From $\tan 2\theta = 2a_{23}/(a_{22} - a_{33}) = \infty$, we get $\theta = \pi/4$. Now define

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{A}_1 = \mathbf{S}_1^{-1}\mathbf{A}\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 5/\sqrt{2} & -3/\sqrt{2} \\ 0 & 5/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 5 & 0 \\ -1/\sqrt{2} & 0 & 3 \end{bmatrix}$$

Now, the largest off-diagonal element in magnitude in $\mathbf{A}_1$ is a_{12} (or a_{13}). We find

$$\tan 2\theta = 2a_{12}/(a_{11} - a_{22}) = -\sqrt{2}/3 \Rightarrow \theta = -0.2203$$

We obtain $\sin\theta$ = -0.2184, $\cos\theta$ = 0.9758. Now, define

$$\mathbf{S}_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.9758 & 0.2184 & 0 \\ -0.2184 & 9578 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_2 = \mathbf{S}_2^{-1}\mathbf{A}_1\mathbf{S}_2$$

$$= \begin{bmatrix} 0.9758 & -0.2184 & 0 \\ 0.2184 & 0.9758 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 5 & 0 \\ -1/\sqrt{2} & 0 & 3 \end{bmatrix} \begin{bmatrix} 0.9758 & 0.2184 & 0 \\ -0.2184 & 0.9758 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9758 & -0.2184 & 0 \\ 0.2184 & 0.9758 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.7971 & 1.1268 & -1/\sqrt{2} \\ -0.4020 & 5.0334 & 0 \\ -0.6900 & -0.1544 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8414 & 0.0002 & -0.6900 \\ 0.0002 & 5.1577 & -0.1544 \\ -0.6900 & -0.1544 & 3 \end{bmatrix}$$

Since $\mathbf{A}_2$ is not a diagonal matrix, we need more iterations. If we neglect the off-diagonal elements, then eigen values after two iterations are obtained as

$$\lambda = 1.84, \quad \lambda = 5.16, \quad \lambda = 3.$$

9. (a) We need an approximation of the form $y = a + bx + cx^2$. We determine a, b, c such that

$$I(a,b,c) = \sum_{i=1}^{6} \left(y_i - a - bx_i - cx_i^2\right)^2 = \text{minimum}$$

We get the normal equations as

$$\frac{\partial I}{\partial a} = -2\sum\left(y_i - a - bx_i - cx_i^2\right) = 0, \quad \frac{\partial I}{\partial b} = -2\sum\left(y_i - a - bx_i - cx_i^2\right)x_i = 0$$

$$\frac{\partial I}{\partial c} = -2\sum\left(y_i - a - bx_i - cx_i^2\right)x_i^2 = 0$$

Hence, we obtain

$$\sum y_i - 6a - b\sum x_i - c\sum x_i^2 = 0, \quad \sum x_i y_i - a\sum x_i - b\sum x_i^2 - c\sum x_i^3 = 0$$

$$\sum x_i^2 y_i - a\sum x_i^2 - b\sum x_i^3 - c\sum x_i^4 = 0$$

From the given data, we obtain

$$\sum x_i = 21, \quad \sum x_i^2 = 91, \quad \sum x_i^3 = 441, \quad \sum x_i^4 = 2275,$$
$$\sum y_i = 3060, \quad \sum x_i y_i = 6450 \quad \text{and} \quad \sum x_i^2 y_i = 17950$$

Substituting these values in the normal equations, we get

$6a + 21b + 91c = 3060$
$21a + 91b + 441c = 6450$
$91a + 441b + 2275c = 17950$

We write these equations as

$a + 3.5b + 15.66667c = 510$
$a + 4.333333b + 21c = 307.142857$
$a + 4.846154b + 25c = 197.252747$

Subtracting, we obtain

$0.833333b + 5.833333c = -202.857143$
$0.512821b + 4c = -109.890110$

Solving these equations, we obtain

$b = -498.434243$, $c = 36.429357$ and $a = 1702.007924$

(b) We have $y' = 1 - 2xy$. Differentiating, we get $y'' = -2y - 2xy'$, $y''' = -4y' - 2x$ $y^{1v} = -6y'' - 2xy'''$. Taylor's series method of order four is given by $y_{n+1} = y_n + hy'_n + h^2 y''_n / 2 + h^3 y'''_n / 6 + h^4 y'^v{}_n / 24, n = 0,1,..........$

We have h= 0.1. We obtain

$n = 0 \quad : \quad x_0 = 0, \quad y_0 = 0, \quad y'_0 = 1, \quad y''_0 = 0, \quad y'''_0 = -4, \quad y^{iv} = 0$

$$y_1 \approx y(0.1) = y_0 + hy'_0 + h^2 y''_0 / 2 + h^3 y'''_0 / 6 + h^4 y_0^{iv} / 24 = 0.099333$$

$n = 1 \quad : \quad x_1 = 0.1, \quad y_1 = 0.099333,$

$y'_1 = 0.980133, \quad y''_1 = -0.394693, \quad y'''_1 = -3.841593, \quad y_1^{iv} = -3.841593$

$$y_2 \approx y(0.2) = y_1 + hy'_1 + h^2 y''_1 / 2 + h^3 y'''_1 / 6 + h^4 y_1^{1v} / 24 = 0.194746$$

10. (a) Taking the limit as $n \to \infty$ and noting that

$\lim_{n\to\infty} x_n = \xi, \ \lim_{n\to\infty} x_{n+1} = \xi$, where ξ is the exact root. We get

(i) $\xi = \frac{1}{2}\xi(1 + \frac{a}{\xi^2}) \Rightarrow \xi^2 = a$, (ii) $\xi = \frac{1}{2}\xi(3 - \frac{\xi^2}{a}) \Rightarrow \xi^2 = a$

Hence, both the methods determine $\sqrt{a}$, where a is a positive real constant.

(i) $\xi + \in_{n+1} = \frac{1}{2}(\xi + \in_n)\left[1 + \frac{\xi^2}{(\xi + \in_n)^2}\right] = \frac{1}{2}(\xi + \in_n)\left[1 + \left(1 + \frac{\in_n}{\xi}\right)^{-2}\right]$

$= \frac{1}{2}(\xi + \in_n)\left[1 + \left\{1 - 2\frac{\in_n}{\xi} + 3\frac{\in_n^2}{\xi^2} - ....\right\}\right] = \frac{1}{2}(\xi + \in_n)$

$\left(2 - 2\frac{\in_n}{\xi} + 3\frac{\in_n^2}{\xi^2} ....\right)$

We obtain, $\in_{n+1} = \in_n^2 / (2\xi) + O(\in_n^3)$. Error constant = $|c| = 1/|2\xi|$ (1)

Hence, the method has second order convergence.

(ii) $\xi + \in_{n+1} = \frac{1}{2}(\xi + \in_n)\left[3 - \frac{1}{\xi^2}(\xi + \in_n)^2\right] = \frac{1}{2}(\xi + \in_n)\left[2 - \frac{2\in_n}{\xi} - \frac{\in_n^2}{\xi^2}\right]$

Simplifying, we obtain, $\in_{n+1} = -3\in_n^2 / (2\xi) + O(\in_n^3)$.

Error constant = $|c^*| = 3/|2\xi|$ (2)

Hence, the method has second order convergence. Comparing (1) and (2), we find that error in the first method is about one third of that in the second method. If we multiply the fist method by 3 and add to the second method, we obtain the method

$$4x_{n+1} = \frac{x_n}{2}\left[3 + \frac{3a}{x_n^2} + 3 - \frac{x_n^2}{a}\right] \quad \text{or} \quad x_{n+1} = \frac{x_n}{8}\left[6 + \frac{3a}{x_n^2} 3 - \frac{x_n^2}{a}\right] \qquad (3)$$

The error of this method is given by

$\in_{n+1}$ = 3 (error in first method) + (error in second method)

$= \left(-3/(2\xi)+3/(2\xi)\right)\in_n^2 + O\left(\in_n^3\right) = O\left(\in_n^3\right)$

Hence, the new method (3) has third order convergence.

11. (a) $TE = f'(x_0) - \left[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right]/(2h)$

Expanding each term in Taylor series about x_0 and simplifying, we get

$TE = h^3 f'''(\xi)/3,\ x_0 < \xi < x_2$. Therefore, $|TE| \le M_3 h^3/3$ where $M_3 = \max|f'''(x)|$.

Let $\in_0, \in_1, \in_2$ be the round-off errors in evaluating f_0, f_1, f_2 respectively. We obtain

$RE = (-3\in_0 + 4\in_1 - \in_2)/2h$. If $\in = \max\left[|\in_0|, |\in_1|, |\in_2|\right]$

then $|RE| \le 8\in/2h = 4\in/h$. We choose h such that

$|RE| = |TE| \Rightarrow 4\in/h = h^3 M_3/3$ which gives $h = (12\in/M_3)^{1/4}$.

(b) From $\mathbf{A} = \mathbf{LL}^T$ where $\mathbf{L} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, we obtain

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 13 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing element by element, we get

First row: $l_{11}^2 = 1 \Rightarrow l_{11} = 1;\quad l_{11}l_{21} = 2 \Rightarrow l_{21} = 2;\ l_{11}l_{31} = 1 \Rightarrow l_{31} = 1$

Second row: $l_{21}^2 + l_{22}^2 = 5 \Rightarrow l_{22}^2 = 1$ or $l_{22} = 1;\quad l_{21}l_{31} + l_{22}l_{32} = 0 \Rightarrow l_{32} = -2$

Third row: $l_{31}^3 + l_{32}^2 + l_{33}^2 = 13 \Rightarrow l_{33}^2 = 8$ or $l_{33} = 2\sqrt{2}$

Hence, we obtain

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 2\sqrt{2} \end{bmatrix}$$

We write the given system of equations **A x = b** as
L L ^{T} **x = b**, or **L**^{T} **x = z** and **L z = b**

From **L z = b,** that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 14 \end{bmatrix}$$

We obtain using forward substitution

$\mathbf{z}_1 = 0$, $\mathbf{z}_2 = -3$, $\mathbf{z}_3 = [\ 14 - \mathbf{z}_1 + 2\ \mathbf{z}_2]/2\ \sqrt{2}\ = 2\sqrt{2}$

From $\mathbf{L}^T\ \mathbf{x} = \mathbf{z}$, that is,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2\sqrt{2} \end{bmatrix}$$

We obtain using back substitution $\mathbf{z} = 1$, $\mathbf{y} = -3 + 2\mathbf{z} = -1$, $\mathbf{x} = -\ 2\mathbf{y} - \mathbf{z} = 1$

Subject **NUMERICAL COMPUTING**

Code **C-09 / T-09 (December 2004)**

1. (a) We are given $f(x) = x^4 - x - 10, x_0 = 1.8, x_1 = 1.9$

We obtain $f_0 = f(x_0) = -1.3024, f_1 = f(x_1) = 1.1321$

Using the secant method, we obtain

First iteration : $x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 1.8535$, $f_2 = f(x_2) = -0.0511$

Second iteration : $x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 1.8555$ **Answer**: **B**

(b) The characteristic equation of the iteration matrix is

$$-\lambda\left[(1/4 - \lambda)^2 - 1/16\right] = 0, \quad \text{or} \quad \lambda^2(\lambda - 1/2) = 0$$

The roots are $\lambda = 0, 0, 1/2$. Spectral radius is 1/2. **Answer**: **C**

(c) Since the points are not equispaced, we use Newton's divided difference interpolation. We have

x	$f(x)$	First *d.d*	Second *d.d*	Third *d.d*
-3	7			
-1	1	-3		
0	1	0	1	
1	3	2	1	0
2	7	4	1	0

Using Newton's divided difference interpolation formula

$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$, we get

$f(x) = 7 + (x+3)(-3) + (x+3)(x+1)(1) = x^2 + x + 1$

Hence, $f(-2) = 3$ **Answer**: **B**

(d) We write the truncation error as

$$TE = f''(x_k) - \frac{1}{12h^2}\left[-30 f(x_k) + 16\{f(x_{k-1}) + f(x_{k+1})\} - \{f(x_{k-2}) + f(x_{k+2})\}\right]$$

Expanding each term in Taylor series about x_k and simplifying, we obtain

$TE = -h^4 f^{(6)}(\xi)/90, \; x_{k-2} < \xi < x_{k+2}$. Hence, $p = 4$. **Answer**: **D**

(e) Make the method exact for $f(x) = 1$, x and x^2. We get

$$f(x)=1: \quad \int_{-1}^{1} dx = 2a+b \qquad \text{or} \qquad 2a+b=2$$

$$f(x)=x: \quad \int_{-1}^{1} x dx = -a+0+a \quad \text{or} \qquad 0=0$$

$$f(x)=x^2: \quad \int_{-1}^{1} x^2 dx = a+0+a \quad \text{or} \qquad 2a = 2/3$$

Hence, we get $a = 1/3,\ b = 4/3$ **Answer**: **D**

(f) Write the integral as $I = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx$, where $f(x) = (1-x^2)^2$.

Using the Gauss-Chebyshev two-point method

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\left[f\left(-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}}\right)\right], \text{ we get } I = \frac{\pi}{2}\left[\frac{1}{4}+\frac{1}{4}\right] = \frac{\pi}{4}$$

Answer: **A**

(g) We need the approximation $f(x) = ax+b$. We determine a and b such that

$I(a,b) = \int_0^1 \left[x^{1/3} - (ax+b)\right]^2 dx =$ minimum. We obtain the normal equations

$$\frac{\partial I}{\partial a} = -2\int_0^1 \left(x^{1/3} - ax - b\right) x\, dx = 0 \qquad \text{or} \qquad \frac{3}{7} - \frac{a}{3} - \frac{b}{2} = 0 \qquad (1)$$

$$\frac{\partial I}{\partial b} = -2\int_0^1 \left(x^{1/3} - ax - b\right) dx = 0 \qquad \text{or} \qquad \frac{3}{4} - \frac{a}{2} - b = 0 \qquad (2)$$

Solving (1) and (2), we get $a = 9/14,\ b = 6/14$ **Answer**: **A**

(h) Euler's method $y_{n+1} = y_n + h\, f(x_n, y_n)$ when applied to the given problem gives

$$y_{n+1} = y_n + h\sqrt{x_n + y_n},\ n = 0, 1, \dots\dots\dots\dots$$

We have h =0.1. We obtain

$n = 0$: $\quad x_0 = 1,\ y_0 = 2,\ \ y_1 = y_0 + h\sqrt{x_0 + y_0} = 2.1732$

$n = 1$: $\quad x_1 = 1.1,\ y_1 = 2.1732,\ \ y_2 = y_1 + h\sqrt{x_1 + y_1} = 2.3541$ **Answer**: **C**

2. (a) Let ξ be the exact root. Since ξ is a root of multiplicity 3, we have

$$f(\xi) = f'(\xi) = f''(\xi) = 0 \text{ and } f'''(\xi) \neq 0$$

Writing $x_{n+1} = \xi + \in_{n+1},\ x_n = \xi + \in_n$ in the given method, we get

$$\xi + \in_{n+1} = \xi + \in_n - \alpha\left[\frac{f(\xi+\in_n)}{f'(\xi+\in_n)}\right]$$

or, $$\in_{n+1} = \in_n - \alpha\left[\frac{\in_n^3 f'''(\xi)/6 + \in_n^4 f^{1v}(\xi)/24 + \dots\dots}{\in_n^2 f'''(\xi)/2 + \in_n^3 f^{1v}(\xi)/6 + \dots\dots}\right]$$

$$= \in_n - \alpha\left[\in_n/3 + \in_n^2 c_4/12 + \dots\dots\right]\left[1 + \{\in_n c_4/3 + \dots\dots\}\right]^{-1}$$

where $c_4 = f^{iv}(\xi)/f'''(\xi)$. Using binomial expansion, we obtain

$$\in_{n+1} = \in_n - \alpha[\in_n/3 + \in_n^2 c_4/12 + ...][1 - \in_n c_4/3 + ...]$$

$$= \in_n - \alpha\left[\frac{1}{3}\in_n - \frac{1}{36}\in_n^2 c_4 + ...\right] \quad = \left(1 - \frac{\alpha}{3}\right)\in_n + \frac{\alpha}{36}\in_n^2 c_4 + O(\in_n^3)$$

Setting the coefficient of $\in_n$ to zero, we get $\alpha = 3$.

The error becomes $\in_n^2 c_4/12 + O(\in_n^3)$.

The method has second order rate of convergence and the error constant is 1/12.

(b) We have $f(x) = x^3 + x^2 + x + 4, \quad f'(x) = 3x^2 + 2x + 1$. We obtain

$\mathbf{n = 0}$: $\quad x_0 = -1.5, \quad f_0 = 1.375, \quad f_0' = 4.75$

$x_0^* = x_0 - f_0/f_0' = -1.7895, \quad f(x_0^*) = -0.3177, \quad x_1 = x_0^* - f(x_0^*)/f_0' = -1.7226.$

$\mathbf{n = 1}$: $\quad x_1 = -1.7226, \quad f_1 = 0.1332, \quad f_1' = 6.4569$

$x_1^* = x_1 - f_1/f_1' = -1.7432, \quad f(x_1^*) = -0.0016, \quad x_2 = x_1^* - f(x_0^*)/f_1' = -1.7430.$

3. (a) We have $f(x, y) = 2x^3 + 4y^3 - 20, \; g(x, y) = 4x^2 + 5y^2 - 21$ and

$$\mathbf{J}_n = \text{Jacobian matrix} = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}_n = \begin{pmatrix} 6x_n^2 & 12y_n^2 \\ 8x_n & 10y_n \end{pmatrix}$$

$$\mathbf{J}_n^{-1} = \frac{1}{D}\begin{bmatrix} 10y_n & -12y_n^2 \\ -8x_n & 6x_n^2 \end{bmatrix}, \quad D = 60x_n^2 y_n - 96x_n y_n^2.$$

Using Newton's method $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \mathbf{J}_n^{-1}\begin{pmatrix} 2x_n^3 + 4y_n^3 - 20 \\ 4x_n^2 + 5y_n^2 - 21 \end{pmatrix}$, we obtain

$\mathbf{n = 0}$: $\quad x_0 = 1.9, \quad y_0 = 0.9, \quad D = 47.196$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1.9 \\ 0.9 \end{pmatrix} - \frac{1}{47.196}\begin{bmatrix} 9 & -9.72 \\ -15.2 & 21.66 \end{bmatrix}\begin{bmatrix} -3.366 \\ -2.51 \end{bmatrix} = \begin{bmatrix} 2.0249 \\ 0.9678 \end{bmatrix}$$

$\mathbf{n = 1}$: $\quad x_1 = 2.0249, \quad y_1 = 0.9678, \quad D = 56.0184$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.0249 \\ 0.9678 \end{pmatrix} - \frac{1}{56.0184}\begin{bmatrix} 9.678 & -11.2396 \\ -16.1992 & 24.6013 \end{bmatrix}\begin{bmatrix} 0.2310 \\ 0.0841 \end{bmatrix} = \begin{bmatrix} 2.0019 \\ 0.9917 \end{bmatrix}$$

(b) The iteration matrix associated with the Gauss-Jacobi iteration method is given by

$$\mathbf{H} = -\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) = -\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}^{-1}\begin{bmatrix} 0 & -6 & 2 \\ -4 & 0 & -1 \\ 1 & -3 & 0 \end{bmatrix}$$

$$=\begin{bmatrix}1/3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/7\end{bmatrix}\begin{bmatrix}0 & 6 & -2\\ 4 & 0 & 1\\ -1 & 3 & 0\end{bmatrix}=\begin{bmatrix}0 & 2 & -2/3\\ 4 & 0 & 1\\ -1/7 & 3/7 & 0\end{bmatrix}$$

The eigen values of $\mathbf{H}$ are obtained from $|\mathbf{H}-\lambda\mathbf{I}| = \begin{vmatrix}-\lambda & 2 & -2/3\\ 4 & -\lambda & 1\\ -1/7 & 3/7 & -\lambda\end{vmatrix} = 0$

We obtain $21\lambda^3 - 179\lambda + 30 = 0$ or $(\lambda+3)(21\lambda^2 - 63\lambda + 10) = 0$

The eigen values of $\mathbf{H}$ are -3, 2.8318, 0.1682

The spectral radius is $\rho(\mathbf{H}) = 3 > 1$. Hence, the method diverges.

4. (a) Let $\mathbf{U} = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$. From $\mathbf{A} = \mathbf{U}\,\mathbf{U}^{\mathbf{T}}$, we get

$$\begin{bmatrix}14 & -7 & 15\\ -7 & 5 & -10\\ 15 & -10 & 25\end{bmatrix} = \begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}\begin{bmatrix}u_{11} & 0 & 0\\ u_{12} & u_{22} & 0\\ u_{13} & u_{23} & u_{33}\end{bmatrix}$$

$$= \begin{bmatrix}u_{11}^2 + u_{12}^2 + u_{13}^2 & u_{12}u_{22} + u_{13}u_{23} & u_{13}u_{33}\\ u_{12}u_{22} + u_{13}u_{23} & u_{22}^2 + u_{23}^2 & u_{23}u_{33}\\ u_{13}u_{33} & u_{23}u_{33} & u_{33}^2\end{bmatrix}$$

Comparing element by element, we get

third column: $u_{33}^2 = 25 \Rightarrow u_{33} = 5$, $u_{23}u_{33} = -10 \Rightarrow u_{23} = -2$

$u_{13}u_{33} = 15 \Rightarrow u_{13} = 5$

second column: $u_{22}^2 + u_{23}^2 = 5 \Rightarrow u_{22}^2 = 1$ or $u_{22} = 1$

$u_{12}u_{22} + u_{13}u_{23} = -7 \Rightarrow u_{12} = -1$

first column: $u_{11}^2 + u_{12}^2 + u_{13}^2 = 14 \Rightarrow u_{11}^2 = 4$ or $u_{11} = 2$

Hence, $\mathbf{U} = \begin{bmatrix}2 & -1 & 3\\ 0 & 1 & -2\\ 0 & 0 & 5\end{bmatrix}$ and $\mathbf{U}^{-1} = \begin{bmatrix}1/2 & 1/2 & -1/10\\ 0 & 1 & 2/5\\ 0 & 0 & 1/5\end{bmatrix}$

Now, $\mathbf{A}^{-1} = (\mathbf{U}\,\mathbf{U}^T)^{-1} = (\mathbf{U}^T)^{-1}\mathbf{U}^{-1} = (\mathbf{U}^{-1})^T\,\mathbf{U}^{-1}$. We obtain

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/10 & 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & -1/10 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/4 & -1/20 \\ 1/4 & 5/4 & 7/20 \\ -1/20 & 7/20 & 21/100 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 25 & 25 & -5 \\ 25 & 125 & 35 \\ -5 & 35 & 21 \end{bmatrix}$$

(b) From the augmented matrix, we have

$$[\mathbf{A}|\mathbf{b}] = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 5 & -3 \\ 3 & 2 & -3 & 6 \end{array}\right) \quad R_1 \approx R_3, \quad \text{(pivoting)}$$

$$\approx \left(\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 2 & 3 & 5 & -3 \\ 1 & 1 & -1 & 2 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1/3 \\ R_3 - R_1/3 \end{array}$$

$$\approx \left(\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 0 & 5/3 & 7 & -7 \\ 0 & 1/3 & 0 & 0 \end{array}\right) \quad R_3 - R_2/5$$

$$\approx \left(\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 0 & 5/3 & 7 & -7 \\ 0 & 0 & -7/5 & 7/5 \end{array}\right)$$

Using back substitution, we get

$$x_3 = -1, \quad x_2 = 3[-7 - 7x_3]/5 = 0, \quad x_1 = [6 - 2x_2 + 3x_3]/3 = 1$$

5. (a) The largest off-diagonal element in magnitude in $\mathbf{A}$ is $a_{13} = 4$.

From $\tan 2\theta = 2a_{13}/(a_{11} - a_{33}) = \infty$, we get $\theta = \pi/4$

Define $\mathbf{S}_1 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ and

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A}\, \mathbf{S}_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ -2 & 5 & -2 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5/\sqrt{2} & -2 & 3/\sqrt{2} \\ -4/\sqrt{2} & 5 & 0 \\ 5/\sqrt{2} & -2 & -3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Now the largest off-diagonal element in magnitude of $\mathbf{A}_1$ is a_{12}.

From $\tan 2\theta = 2a_{12}/(a_{11}-a_{22}) = -\infty$, we get $\theta = -\pi/4$

Define $\mathbf{S}_2 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$\mathbf{A}_2 = \mathbf{S}_2^T \mathbf{A}_1 \mathbf{S}_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (5+2\sqrt{2})/\sqrt{2} & (5-2\sqrt{2})/\sqrt{2} & 0 \\ -(5+2\sqrt{2})/\sqrt{2} & (5-2\sqrt{2})/\sqrt{2} & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 5+2\sqrt{2} & 0 & 0 \\ 0 & 5-2\sqrt{2} & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Hence, the eigen values of $\mathbf{A}$ are $5+2\sqrt{2}$, $5-2\sqrt{2}$ and -3.

(b) We have $\mathbf{A} - 3\mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{B} = (\mathbf{A} - 3\mathbf{I})^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

Starting with $\mathbf{V}_0 = [1, \quad -1, \quad 1]^T$, we get

$\mathbf{Y}_1 = \mathbf{B}\,\mathbf{V}_0 = [-2, \quad 3, \quad -2]^T; m_1 = 3;\ \mathbf{V}_1 = \mathbf{Y}_1 / m_1 = [-2/3, \quad 1, \quad -2/3]^T$

$\mathbf{Y}_2 = \mathbf{B}\,\mathbf{V}_1 = [5/3, \quad -7/3, \quad 5/3]^T; m_2 = 7/3; \mathbf{V}_2 = \mathbf{Y}_2 / m_2 = [5/7, \quad -1, \quad 5/7]^T$

$\mathbf{Y}_3 = \mathbf{B}\,\mathbf{V}_2 = [-12/7, \quad 17/7, \quad -12/7]^T$

After three iterations, we obtain the ratios

$$|\mu| = \frac{(\mathbf{Y}_3)_r}{(\mathbf{V}_2)_r} = \left[\left|\frac{-12}{5}\right|, \left|\frac{-17}{5}\right|, \left|\frac{-12}{5}\right|\right] = [2.4, \quad 3.4, \quad 2.4]$$

We take $|\mu| \approx 2.4$ (we need more iterations for more accurate results.)

Hence, the largest eigen value in magnitude of $\mathbf{B}$ is $|\mu| = 2.4$

The eigen value nearest to 3 of the matrix $\mathbf{A}$ is

$|\lambda| = 3 \pm (1/|\mu|) = 3 \pm (1/2.4)$ or $\lambda = 3.4167, \ \lambda = 2.5833$

Since $\lambda = 2.5833$ satisfies the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$, more accurately, the nearest eigen value to 3 is 2.5833.

6. (a) We obtain from the given matrix

$$\tan\theta = a_{13}/a_{12} = 4/\sqrt{2} = 2\sqrt{2}$$

Therefore $\sin\theta = 2\sqrt{2}/3$ and $\cos\theta = 1/3$

Define $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$ and

$$\mathbf{A}_1 = \mathbf{S}_1^T \mathbf{A}\,\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 2\sqrt{2}/3 \\ 0 & -2\sqrt{2}/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 & \sqrt{2} & 4 \\ \sqrt{2} & 6 & \sqrt{2} \\ 4 & \sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 2\sqrt{2}/3 \\ 0 & -2\sqrt{2}/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 & 3\sqrt{2} & 0 \\ \sqrt{2} & 10/3 & -11\sqrt{2}/3 \\ 4 & 5\sqrt{2}/3 & -2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3\sqrt{2} & 0 \\ 3\sqrt{2} & 10/3 & -5\sqrt{2}/3 \\ 0 & -5\sqrt{2}/3 & 14/3 \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & 0 \\ c_1 & b_2 & c_2 \\ 0 & c_2 & b_3 \end{bmatrix}$$

which is the required tri-diagonal form. Using Sturms sequence, we obtain

$f_0 = 1, \quad f_1 = \lambda - b_1 = \lambda - 2$

$f_2 = (\lambda - b_2) f_1 - c_1^2 f_0 = (\lambda - 10/3)(\lambda - 2) - 18 = \lambda^2 - 16\lambda/3 - 34/3$

$f_3 = (\lambda - b_3) f_2 - c_2^2 f_1 = (\lambda - 14/3)(\lambda^2 - 16\lambda/3 - 34/3) - 50(\lambda - 2)/9$

$= \lambda^3 - 10\lambda^2 + 8\lambda + 64.$

The characteristic equation of the given matrix is $f_3 = \lambda^3 - 10\lambda^2 + 8\lambda + 64 = 0$

(b) From the given matrix, we write

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 4 & 1 & 1 & 1 & 0 & 0 \\ 1 & 4 & -2 & 0 & 1 & 0 \\ 3 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_2 - R_1/4 \\ R_3 - 3R_1/4 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 15/4 & -9/4 & -1/4 & 1 & 0 \\ 0 & 5/4 & -19/4 & -3/4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_1 - 4R_2/15 \\ R_3 - R_2/3 \end{array}$$

$$\approx \left[\begin{array}{ccc|ccc} 4 & 0 & 8/5 & 16/15 & -4/15 & 0 \\ 0 & 15/4 & -9/4 & -1/4 & 1 & 0 \\ 0 & 0 & -4 & -2/3 & -1/3 & 1 \end{array}\right] \quad \begin{array}{l} \\ R_1 + 2R_3/5 \\ R_2 - 9R_3/16 \end{array}$$

$$\approx \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 4/5 & -2/5 & 2/5 \\ 0 & 15/4 & 0 & 1/8 & 19/16 & -9/16 \\ 0 & 0 & -4 & -2/3 & -1/3 & 1 \end{array}\right) \quad R_1/4,\ 4R_2/15,\ -R_3/4$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 & -1/10 & 1/10 \\ 1/30 & 19/60 & -9/60 \\ 1/6 & 1/12 & -1/4 \end{bmatrix}$$

$$\text{Hence, } \mathbf{A}^{-1} = \begin{bmatrix} 1/5 & -1/10 & 1/10 \\ 1/30 & 19/60 & -9/60 \\ 1/6 & 1/12 & -1/4 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 12 & -6 & 6 \\ 2 & 19 & -9 \\ 10 & 5 & -15 \end{bmatrix}$$

7. (a) We need to determine a and b such that

$I(a,\ b) = \int_1^4 \left[x^2 - \left(a\sqrt{x} + b/x\right)\right]^2 dx = \text{minimum}$. We obtain the normal equations

$$\frac{\partial I}{\partial a} = -2\int_1^4 \left(x^2 - a\sqrt{x} - b/x\right)\sqrt{x}\, dx = 0 \quad \text{or} \quad \frac{254}{7} - \frac{15a}{2} - 2b = 0 \qquad (1)$$

$$\frac{\partial I}{\partial b} = -2\int_1^4 \left(x^2 - a\sqrt{x} - \frac{b}{x}\right)\frac{1}{x}\, dx = 0, \quad \text{or} \quad \frac{15}{2} - 2a - \frac{3}{4}b = 0 \qquad (2)$$

Solving (1) and (2), we get $a = 684/91,\ b = -914/91$.

(b) Newton's backward difference interpolation is given by

$$f(x) = f_n + \frac{(x - x_n)}{h}\nabla f_n + \frac{(x - x_n)(x - x_{n-1})}{2!h^2}\nabla^2 f_n + \frac{(x - x_n)(x - x_{n-1})(x - x_{n-2})}{3!h^3}\nabla^3 f_n + \ldots\ldots$$

Substituting $x - x_n = hs$, we get

$$f(x_n + hs) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \ldots\ldots\ldots\ldots$$

We obtain

$$\frac{df}{ds} = \frac{df}{dx}.\frac{dx}{ds} = \nabla f_n + \frac{2s+1}{2!}\nabla^2 f_n + \frac{3s^2 + 6s + 2}{3!}\nabla^3 f_n + \ldots\ldots\ldots$$

Now $dx/ds = h$ and for $x = x_n$, we get $s = 0$. We obtain

$$f'(x_n) = \frac{df}{dx} = \frac{1}{h}\left[\nabla f_n + \frac{1}{2}\nabla^2 f_n + \frac{1}{3}\nabla^3 f_n + \ldots\ldots\ldots\ldots\right] \qquad (1)$$

We now construct the backward difference table from the given data. We get for h = 1

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
1	1			
2	3	2		
3	7	4	2	

4	13	6	2	0
5	21	8	2	0

Substituting in (1) for x_n= 5, we obtain $f'(5) = [8 + (2/2)] = 9$

8. (a) Using Lagrange interpolation formula and the given data, we get

$$p(x) = \frac{(x+1)x(x-1)}{(-3+1)(-3)(-3-1)}(-29) + \frac{(x+3)x(x-1)}{(-1+3)(-1)(-1-1)}(-1)$$

$$+ \frac{(x+3)(x+1)(x-1)}{(3)(1)(-1)}(1) + \frac{(x+3)(x+1)x}{(1+3)(1+1)(1)}(3)$$

$$= \frac{29}{24}(x^3 - x) - \frac{1}{4}(x^3 + 2x^2 - 3x) - \frac{1}{3}(x^3 + 3x^2 - x - 3) + \frac{3}{8}(x^3 + 4x^2 + 3x)$$

$$= x^3 + x + 1$$

$$f(0.5) \approx p(0.5) = (0.5)^3 + (0.5) + 1 = 1.625$$

(b) we have $f[x_0, x_1] = \frac{1}{x_1 - x_0}[f(x_1) - f(x_0)] = \frac{1}{x_1 - x_0}\left[\frac{1}{x_1} - \frac{1}{x_0}\right] = -\frac{1}{x_0 x_1} = \frac{(-1)^{-1}}{x_0 x_1}$

We shall now show by induction that $f[x_0, x_1 \ldots\ldots, x_n] = (-1)^n / (x_0 x_1 \ldots\ldots x_n)$

The result is true for n = 1. Assume that the result is true for $n = k$, that is

$f[x_0, x_1 \ldots\ldots, x_k] = (-1)^k / (x_0 x_1 \ldots\ldots x_k)$. Then for $n = k + 1$, we have

$$f[x_0, x_1 \ldots\ldots, x_k, x_{k+1}] = \frac{1}{x_{k+1} - x_0}\{f[x_1, x_2, \ldots\ldots x_{k+1}] - f[x_0, x_1, \ldots\ldots, x_k]\}$$

$$= \frac{1}{x_{k+1} - x_0}\left[\frac{(-1)^k}{x_1 x_2 \ldots\ldots x_{k+1}} - \frac{(-1)^k}{x_0 x_1 \ldots\ldots x_k}\right] = \frac{(-1)^{k+1}}{x_0 x_1 \ldots\ldots x_{k+1}}$$

Hence, the result is true for any k.

9. (a) We can write $TE = f'(x_0) - [f(x_0 + h) - f(x_0)]/h$

$$= f'(x_0) - [hf'(x_0) + h^2 f''(x_0)/2 + h^3 f'''(x_0)/6 + \ldots\ldots]/h$$

$$= -hf''(x_0)/2 - h^2 f'''(x_0)/6 - \ldots\ldots$$

Hence, we can write the error term as

$TE = c_1 h + c_2 h^2 + \ldots\ldots + c_p h^p + O(h^{p+1})$, where c_i' s are independent of h.

Let $g(x) = f'(x_0)$ be the exact quantity and $g(h/2^r)$ denote the approximate value of $g(x)$ obtained by using the given method with step length $h/2^r$, r =0, 1, ………….

Thus we can write

$$g(h) = g(x) + c_1 h + c_2 h^2 + c_3 h^3 + \dots\dots\dots$$

$$g(h/2) = g(x) + c_1 h/2 + c_2 h^2/4 + c_3 h^3/8 + \dots\dots\dots$$

$$g(h/2^2) = g(x) + c_1 h/4 + c_2 h^2/16 + c_3 h^3/64 + \dots\dots\dots\dots$$

$\vdots$

Eliminating c_1 from the above equations, we obtain

$$g^{(1)}(h) = \frac{2g(h/2) - g(h)}{2-1} = g(x) - \frac{c_2 h^2}{2} - \frac{3}{4} c_3 h^3 - \dots\dots\dots\dots$$

$\vdots$

$$g^{(1)}(h/2) = \frac{2g(h/2^2) - g(h/2)}{2-1} = g(x) - \frac{c_2 h^2}{8} - \frac{3}{32} c_3 h^3 - \dots\dots\dots\dots$$

$\vdots$

Eliminating c_2 from the above equations, we get

$$g^{(2)}(h) = \frac{4g^{(1)}(h/2) - g(h)}{4-1} = \frac{2^2 g^{(1)}(h/2) - g^{(1)}(h)}{2^2 - 1} + O(h^3)$$

Thus the successive higher order results can be obtained from the formula

$$g^{(m)}(h) = \frac{2^m g^{(m-1)}(h/2) - g^{(m-1)}(h)}{2^m - 1}, \; m = 1,\; 2,\; \dots\dots\dots,\; g^{(0)}(h) = g(h)$$

Using the given formula and the given data, we obtain

$h = 4$: $\quad f'(1) = [f(5) - f(1)]/4 = (125 - 1)/4 = 31$

$h = 2$: $\quad f'(1) = [f(3) - f(1)]/2 = (27 - 1)/2 = 13$

$h = 1$: $\quad f'(1) = [f(2) - f(1)] = (8 - 1) = 7$

We obtain the following Richardson's table

h	$O(h)$	$O(h^2)$	$O(h^3)$
4	31		
2	13	-5	
1	7	1	3

Hence, the best value of $f'(1)$ is 3.

(b). Truncation error of the given method is given by

$$TE = f'(x_0) - [-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)]/(2h)$$

Expanding each term in Taylor series about x_o and simplifying, we get

$TE = \frac{h^3}{3} f'''(\xi), \quad x_0 < \xi < x_2$. Hence $|TE| \le \frac{h^2}{3} M_3$, where $M_3 = \max_{x \le x \le x_2} |f'''(x)|$

If $\in_0, \in_1, \in_2$ are round off errors in evaluating $f(x_0), f(x_0 + h), f(x_0 + 2h)$ respectively,

then we get $RE = (-3\in_0 + 4\in_1 - \in_2)/2h$

Let $\in = \max[|f_0|, |\in_1|, |\in_2|]$. Then $|RE| \le 8\in/(2h) = 4\in/h$

We choose h such that $h^2 M_3/3 + 4\in/h = \text{minimum}$

Differentiating with respect to h and equating it to zero, we get

$$2hM_3/3 - 4\in/h^2 = 0 \text{ or } h = (6\in/M_3)^{1/3}$$

10. (a) Make the method exact for $f(x) = 1$, x and x^2. We get

$$f(x) = 1: \quad \int_{-1}^{1} dx - (\lambda_0 + \lambda_1 + \lambda_2) = 0$$

$$f(x) = x: \quad \int_{-1}^{1} x dx - \left[\lambda_0\left(-\sqrt{3/5}\right) + \lambda_2\left(\sqrt{3/5}\right)\right] = 0$$

$$f(x) = x^2: \quad \int_{-1}^{1} x^2 dx - \left[\lambda_0\left(-\sqrt{3/5}\right)^2 + \lambda_2\left(\sqrt{3/5}\right)^2\right] = 0$$

From the above equations, we obtain

$\lambda_0 + \lambda_1 + \lambda_2 = 2$, $(\lambda_0 - \lambda_2) = 0$, $(\lambda_0 + \lambda_2) = 10/9$

Solving these equations, we get $\lambda_0 = \lambda_2 = 5/9$ and $\lambda_1 = 8/9$

It can be verified that the method produces exact results for $f(x) = x^3, x^4$ and x^5.

The error term is given by Error = $cf^{vi}(\xi)/6!$, $\quad -1 < \xi < 1$, where

$$c = \int_{-1}^{1} x^6 dx - \left[\lambda_0\left(-\sqrt{3/5}\right)^6 + \lambda_2\left(\sqrt{3/5}\right)^6\right] =$$

$$\frac{2}{7} - (\lambda_0 + \lambda_2)\frac{27}{125} = \frac{2}{7} - \frac{10}{9}\left(\frac{27}{125}\right) = \frac{8}{175}$$

Hence, Error = $\frac{8}{175}\left(\frac{1}{720}\right) f^{(6)}(\xi) = \frac{1}{15750} f^{(6)}(\xi)$.

(b). Using the Trapezoidal rule, we get

$h = 1$: $\quad x_0 = 1, \quad x_1 = 2, \quad f_0 = f(x_0) = 1, \quad f_1 = f(x_1) = 1/3$

$$I = h[f_0 + f_1]/2 = 0.666667$$

$h = 1/2$: $\quad x_0 = 1, \quad x_1 = 3/2, \quad x_2 = 2 \quad f_0 = 1, \quad f_1 = 0.571429, \quad f_2 = f(x_2) = 1/3$

$$I = h[f_0 + 2f_1 + f_2]/2 = 0.619048$$

$h = 1/4$: $\quad x_0 = 1, \quad x_1 = 5/4, \quad x_2 = 6/4 \quad x_3 = 7/4 \quad x_4 = 2$

$f_0 = 1, \quad f_1 = 0.761905, \quad f_2 = 0.571429$

$f_3 = 0.432432, \quad f_4 = 1/3$

$$I = h[f_0 + 2(f_1 + f_2 + f_3) + f_4]/2 = 0.608108$$

Using Romberg integration $I^{(m)}(h) = \frac{4^m I^{(m-1)}(h/2) - I^{(m-1)}(h)}{4^m - 1}$, $m = 1, 2, \ldots\ldots\ldots$

we obtain the following Romberg table

h	$o(h^2)$	$o(h^4)$	$o(h^6)$
1	0.666667		
½	0.619048	0.603175	
¼	0.608108	0.604461	0.604547

The improved value of I is $I = 0.604547$

11. (a) We have $y' = x + y^2$, $y'' = 1 + 2y\, y'$, $y''' = 2(y')^2 + 2yy''$.

Third order Taylor series method is given by

$$y_{n+1} = y_n + hy'_n + h^2 y''_n / 2 + h^3 y'''_n / 6, \; n = 0, 1, \ldots\ldots\ldots..$$

We have $h = 0.2$, $x_0 = 1$, $y_0 = 1$. We obtain

$n = 0$: $y_0 = 1$, $y'_0 = 2$, $y''_0 = 5$, $y'''_0 = 18$

$$y_1 \approx y(1.2) = y_0 + hy'_0 + h^2 y''_0 / 2 + h^3 y'''_0 / 6$$

$$= 1 + 0.2(2) + (0.2)^2 (5)/2 + (0.2)^3 (18)/6 = 1.524$$

Now $x_1 = 1.2$, $y_1 = 1.524$, $y'_1 = 3.522576$

$y''_1 = 11.736812$, $y'''_1 = 60.590886$

$$y_2 \approx y(1.4) = y_1 + hy'_1 + h^2 y''_1 / 2 + h^3 y'''_1 / 6$$

$$= 1.524 + 0.2(3.522576) + \frac{(0.2)^2}{2}(11.736812) + \frac{(0.2)^3}{6}(60.590886) = 2.544039$$

(b). We have $x_0 = 1$, $y_0 = 2$, $f(x, y) = \sqrt{x + 2y}$, $h = 0.1$

Using the classical fourth order Runge-Kutta method, we get

$$k_1 = hf(x_0, y_0) = 0.1 f(1,2) = 0.223607$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2) = 0.1 f(1.05, 2.111804) = 0.229643$$

$$k_3 = hf(x_0 + h/2, y_0 + k_2/2) = 0.1 f(1.05, 2.114822) = 0.229775$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(1.1, 2.229775) = 0.235786$$

$$y_1 \approx y(1.1) = y_0 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 2.29705$$

Now, $x_1 = 1.1$, $y_1 = 2.229705$. we get

$$k_1 = hf(x_1, y_1) = 0.1 f(1.1, 2.229705) = 0.235784$$

$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.1 f(1.15, 2.347597) = 0.241768$$

$$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.1 f(1.15, 2.350589) = 0.241892$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 f(1.2, 2.471597) = 0.247855$$