Q2 (a) If $\sin y = x\sin(a+y)$**, prove that** $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$**.**

Answer

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

$$\Rightarrow \text{If } \sin y = x\sin(a+y)$$

Differtiating both sides w.r.t x, we get

$$\Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}(x\sin(a+y))$$

$$\Rightarrow \cos y\frac{dy}{dx} = 1.\sin(a+y) + x\cos(a+y).\frac{dy}{dx}$$

$$\Rightarrow \cos y\frac{dy}{dx} - x\cos(a+y).\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow (\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow [\sin y = x\sin(a+y), x\frac{\sin y}{\sin(a+y)}]$$

$$\Rightarrow (\frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\sin(a+y)})\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \frac{\sin(a+y-y)}{\sin(a+y)}\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

=> Alternative - - - -

=> Since we have $\sin y = x\sin(a+y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

=> Defferentiating both sides w.r.t' y'

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(a+y)\cos y - \cos(a+y)\sin y}{\sin^2(a+y)}$$

$$\Rightarrow \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = 1/\frac{dy}{dx} = \frac{1}{\sin a/\sin^2(a+y)} = \frac{\sin^2(a+y)}{\sin a}$$

Q 2 (B) Prove that the straight line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-x/a}$ at the point where the curve crosses the axis of y.

Answer

$$y = be^{-x/a}$$

$$=> \frac{dy}{dx} = be^{-x/a}(-1/a)$$

$$=> \frac{-b}{a}e^{-x/a}$$

$$=> (\frac{dy}{dx})_{(0,\,b)} = -\frac{-b}{a}e^{-0/a} = \frac{-b}{a}$$

so, the equation of the tangent at (0, b) is

$$=> \text{ y - b} = -\frac{-b}{a}(x-0)$$

$$=> ay - ab = -bx$$

$$=> bx + ay = ab$$

Dividing both side unit'ab'

$$=> \frac{dx}{ab} = \frac{ay}{ab} = \frac{ab}{ab}$$

$$=> \frac{x}{a} + \frac{y}{b} = 1$$

.

Hence $\frac{x}{a}+\frac{y}{b}=1$ touch as the curve $y = be^{-x/a} at(0,b)$

Q3 (a) Evaluate $\int \frac{1}{\sqrt{x(1-2x)}}dx$

Answer

$$\int \frac{1}{\sqrt{x(1-2x)}}dx = \int \frac{1}{\sqrt{x-2x^2}}dx$$

$$=> \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{-\left(x^2-\frac{x}{2}+(\frac{1}{4})^2-(\frac{1}{4})^2\right)}}dx$$

$$=> \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left((\frac{1}{4})2-(x\frac{1}{4})^2\right)}}dx$$

$$=> \frac{1}{\sqrt{2}}\sin^{-1}(\frac{x-1/4}{1/4})+c$$

$$=> \frac{1}{\sqrt{2}}\sin^{-1}(4x-1)+c$$

Q3 (b) Evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$

Answer

$$\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$$

$=>$ Put $\sin\theta = t,$ so that $\cos d\theta = dt$

when $\theta = 0, t = \text{Sin}0 = 0$

when $\theta = \pi/2, t = \sin 0 = 0$

when $\theta = \pi/2, t = \sin \pi/2 = 1$

$$=> \text{I} = \int_0^1 \frac{1}{(1+t)(2+t)} dt$$

$$=> \frac{1}{(1+t)(2+t)} = \frac{A}{(1+t)} + \frac{B}{(2+t)}$$

$=> 1 = a(2+t) + B(1+t)$

$=>$ put t = -1, we get

$1 = \text{A}(2 - 1)$

$\text{A} = 1$

$=>$ put t = -2, we get

$1 = \text{B}(1 - 2)$

$\text{B} = -1$

$$=> \frac{1}{(1+t)(2+t)} = \frac{1}{(1+t)} - \frac{1}{(2+t)}$$

$$=> \text{I} = \int_0^1 (\frac{1}{(1+t)} - \frac{1}{(2+t)}) dt$$

$=> \text{I}(\log(1+\text{t}))^2{}_0 - (\log(2+\text{t}))^2{}_0$

$=> (\log(1+1) - \log(1+0)) - (\log(2+1) - \log(2+0)$

$=> (\log 2 - \log 1) - (\log 3 - \log 2)$

$=> \log 2 - \log 3 + \log 2$

$=> 2\log 2 - \log 3$

$=> \log 2^2 - \log 3$

$=> \log 4 - \log 3$

$= \log(4/3)$

Q4 (a) Prove that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

Answer

LHS

$$let\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$\text{let } \Delta => \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 - R_1 + R_2 + R_3$, *we get*

$$\Delta => \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$=> (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

=> *A*pplying $C_3 - C_3 - C_1, C_2 - C_2 - C_1$

$$=> (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$$

$$=> (a+b+c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix} (\text{taking}(a+b+c) \text{ common from } C_2 \,\&\, C_3)$$

Expending from R_1

$$=> (a+b+c)^3, 1\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = (a+b+c)^3 = R.H.S$$

Q4 (b) Apply Cramer's rule to solve the following system of linear equations

$$x + y + z = -1$$
$$x + 2y + 3z = -4$$
$$x + 3y + 4z = -6$$

Answer

$x + y + z = -1$

$x + 2y + 3z = -4$

$x + 3y + 4z = -6$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

Expanding by 'R_1'

$\Rightarrow 1(8-9) - 1(4-3) + 1(3-2)$

$\Rightarrow (-1) - 1(1) + 1(1)$

$\Rightarrow -1 - 1 + 1(1)$

$$\Rightarrow \Delta_1 = \begin{vmatrix} -1 & 1 & 1 \\ -4 & 2 & 3 \\ -6 & 3 & 4 \end{vmatrix}$$

$\Rightarrow$ *E*xpanding by R_2

$\Rightarrow 1(8-9) - 1(16+18) + 1(-12+12)$

$\Rightarrow -1(-1) - 1(2) + 0$

$\Rightarrow 1 - 2 = -1$

$$\Rightarrow \Delta_2 = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 4 & 3 \\ 1 & -6 & 4 \end{vmatrix}$$

Expanding by R_1

$\Rightarrow 1(-16+18) + 1(4-3) + 1(-6+4)$

$\Rightarrow 1(2) + 1(1) + 1(-2)$

$\Rightarrow 2 + 1 - 2 = 1$

$$\Rightarrow \ \& \ \Delta_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -4 \\ 1 & 3 & -6 \end{vmatrix}$$

Expanding by R_1

$\Rightarrow 1(-12+12) - 1(-6+4) - 1(3-2)$

$\Rightarrow 1(0) - 1(-2) - 1(1)$

$\Rightarrow 2 - 1 = 1$

$\Rightarrow x = \Delta_1/\Delta = -1/-1 = 1$

$\Rightarrow y = \Delta_2/\Delta = 1/-1 = -1$

$\Rightarrow z = \Delta_3/\Delta = 1/-1 = -1$

Q5 (a) Solve $x\dfrac{dy}{dx}+\cot y=0$**, given that** $y=\dfrac{\pi}{4}$ **when** $x=\sqrt{2}$

Answer

$x\dfrac{dy}{dx}+\cot y=0$

$x\dfrac{dy}{dx}=-\cot y$

by Separating the variables, we have

$=> \dfrac{dx}{x}=-\tan y\,dy$

Interfacing both sides :–

$\int\dfrac{1}{x}\,dx=-\int \tan y\,dy$

$\log|x|=\log|\cos y|+\log|c|$

$\log|x|=\log|c\cos y|$

Taking antilog both sides :–

$x=c\cos y$

when $y=\pi/4,\ x=\sqrt{2}$

$\sqrt{2}=c\cos y\,\pi/4$

$\sqrt{2}=c.\ 1/\sqrt{2}$

$c=2$

$x=2\cos y$

Q5 (b) Solve $\dfrac{dy}{dx}+y\sec x=\tan x$

Answer

the given differential equation is

$\dfrac{dy}{dx}+\sec y=\tan x$................................(1)

This is a linear equation of the form $\dfrac{dy}{dx}+py=Q$, where

$P=\sec x,\ Q=\tan x$

=> we have $\text{I.F}=e^{\int\sec x\,dx}=e^{\log(\sec x+\tan x)}$

$=\sec x+\tan x$

multiplying both sides of (1) by sec x + tanx we get

$\dfrac{dy}{dx}(\sec x+\tan x)+\sec x(\sec x+\tan x)y=(\tan x(\sec x+\tan x)$

integrating w.r.t x, we get

$y(\sec+\tan x)=\int\tan x(\tan x+\sec x)\,dx$

(using $y\,(\text{I.F})=\int Q(I.F)\,dx+c$

$=\int\sec x\tan x\,dx+\int\tan^2 x\,dx+c$

$=y(\sec x+\tan x)=\sec x+\tan x-x+c$

$=$

Q 6(a) Prove that $\cos^2 A+\cos^2(A+120^\circ)+\cos^2(A-120^\circ)=\frac{3}{2}$

Answer

$$LHS$$
$$=> \cos^2 A+\cos^2(A+120^\circ)+\cos^2(A-120^\circ)$$
$$=> \frac{1+\cos 2A}{2}+\frac{1+\cos(2A+240)}{2}+\frac{1+\cos(2A-240)}{2}$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A+\cos(2A+240^\circ)+\cos(2A-240^\circ))$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A+2\cos\frac{4A}{2}\cos\frac{480^\circ}{2})$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A+2\cos 2A\cos 240^\circ)$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A+2\cos 2A\cos(180^\circ+60^\circ))$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A-2\cos 2A\cos 60^\circ)$$
$$=> \frac{3}{2}+\frac{1}{2}(\cos 2A-2\cos 2A(\frac{1}{2}))$$
$$=> \frac{3}{2}+\frac{1}{2}(0)=\frac{3}{2}\ RHS$$

Q 6(b) If $A+B+C=\pi$**, prove that** $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

Answer

We have $A+B+C=\pi$
$$=> \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$
$$=> \frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$$
Taking 'tan' both sides :-
$$=> \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{c}{2}=\frac{1}{\tan\frac{c}{2}}$$

cross multipliction, we get
$$=> \tan\frac{A}{2}\tan\frac{c}{2}+\tan\frac{B}{2}+\tan\frac{c}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$
$$=> \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{c}{2}+\tan\frac{c}{2}\tan\frac{A}{2}=1$$
Dividing both sides with $an\frac{A}{2}\tan\frac{B}{2}\tan\frac{c}{2}$ we get
$$=> \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{c}{2}}+\frac{\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{c}{2}}=\frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{c}{2}}$$
$$=> \frac{1}{\tan\frac{c}{2}}+\frac{1}{\tan\frac{A}{2}}\frac{1}{\tan\frac{B}{2}}=\frac{1}{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{c}{2}}$$
$$=> Cot\frac{A}{2}+Cot\frac{B}{2}+Cot\frac{C}{2}=Cot\frac{A}{2}Cot\frac{B}{2}Cot\frac{C}{2}$$

Q7 (a) Find the term independent of 'x' in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

Answer

In the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$, Trh *is equal* to

$$9_{C_r}\left(\frac{3x^2}{2}\right)^{9-r}\left(-\frac{1}{3x}\right)^r$$

$$\Rightarrow 9_{C_r}\left(\frac{3}{2}\right)^{9-r}x^{18-2r}\left(-\frac{1}{3x}\right)^r x^{-r}$$

$$\Rightarrow 9_{C_r}\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

let Trh be the term independent of 'x'

$\Rightarrow 18 - 3r = 0$ or $r = 6$

Required term

$$Tr_h = T_{6+1} = 9_{C_6}\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 x^{18-3(6)}$$

$$\Rightarrow 84\left(\frac{27}{8}\right)\left(\frac{1}{729}\right)x^0 = \frac{7}{18}$$

Q7 (b) If the 5^{th} term of a G.P. is 16 and the 10^{th} term is $\frac{1}{2}$, find the G.P. Also find its 15^{th} term.

Answer

let 'a' be the first term and 'r' be the common ratio of the G.P

$T_n = ar^{n-1}, n \in N$

we have $T_5 = ar^{5-1} = 16$

$\& \; T_{10} = ar^{10-1} = \frac{1}{2}$

$\Rightarrow ar^4 = 16 \;\; \& \; ar^9 = \frac{1}{2}$

$\Rightarrow \frac{ar^9}{ar^4} = \frac{1/2}{16} = \frac{1}{32}$

$\Rightarrow r^5 = \frac{1}{32} = (\frac{1}{2})^5$

$\Rightarrow r = \frac{1}{2}$

$\Rightarrow ar^4 = 16$

$\Rightarrow a(\frac{1}{2})^4 = 16;$

$\Rightarrow a = 16 * 16 = 256$

The G.P. is 256, 256 (1/2), 256$(1/2)^2$…………… or 256, 128, 64 ……………...

Also T15 = ar^{14} = 256 $(1/2)^{14}$ = 1/64

Q8 (a) A line passes through (3, 4) and the sum of its intercepts on the axis is 14, find the equation of the line.

Answer

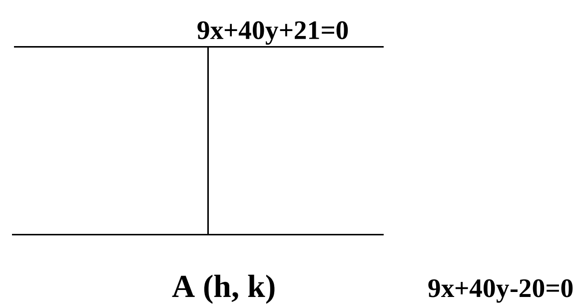
Let x intercept of the line = a
y intercept of the line =14- a

.:. The equation on of a line in intercept form is x/a+y/14-a=1………………………(1)
This line passes through (3, 4)

$$=> \frac{3}{a} + \frac{4}{14-a} = 1$$
$$=> \frac{3(14-a)+4a}{a(14-a)} = 1$$
$$=> 3(14-a)+4a = a(14-a)$$
$$=> 42-3a+4a = 14a-a^2$$
$$=> a^2 - 13a + 42 = 0$$
$$=> a^2 - 6a - 7a + 42 = 0$$
$$=> a(a-6) - 7(a-6) = 0$$
$$=> (a-6)(a-7) = 0$$
$$=> a = 6,7$$
$$CASE - 1$$
$$=> a = 6$$
$$\text{then b} = 14 - 6 = 8$$
$$.:. \text{ Equation of line will be}$$
$$=> \frac{x}{6} + \frac{y}{8} = 1$$
$$CASE - 2$$
$$=> a = 7$$
$$\text{then b} = 14 - 7 = 7$$
$$.:. \text{ Equation of line will be}$$
$$=> \frac{x}{7} + \frac{y}{7} = 1$$

Q8 (b) Find the distance between the lines $9x + 40y - 20 = 0$ **and** $9x + 40y + 21 = 0$

Answer



The given lines are

$9x + 40y - 20 = 0$ ....(1)

$9x + 40y + 21 = 0$ ....(2)

Scope of line (1) = scop of line (2),. :. The lines are parallel let A(h,k) lies on the line 9h+40k-20=0 .:. 9h+40k-20=0............................(3)

.:. Distance from A (h,k) on line 9h+40k-21=0

is $=> \left|\frac{9h+40k+21}{\sqrt{81+1600}}\right| = \left|\frac{20+21}{\sqrt{1681}}\right| = \left|\frac{41}{41}\right|$

Q9 (a) Find the equation of the circle whose centre is the point $(1,-2)$ and which passes through the centre of the circle $x^2 + y^2 + 2y - 3 = 0$

Answer Let the equation of the required circle be

$=> x^2 + y^2 + 2gx + 2fy + c = 0$....................(1)

$=> Centre\ is = (-g,-f) = (1,-2)$

$.:. -g = 1, -f = -2$

$g = -1, f = 2$

$.:.(1) => x^2 + y^2 - 2x + 4y + c = 0$...............(2)

the given circle is $x^2 + y^2 + 2y - 3 = 0$.......................(3)

$=> x^2 + y^2 + 2(0)x + 2(1)y - 3 = 0$

.:. centre of this circle is (0,-1)

the required circle () passes through (0,-1)

$(0)^2 + (-1)^2 - 2(0) + 4(-1) + c = 0$

$0 + 1 - 0 - 4 + c = 0$

$c = 3$

.:. the equation of required circle is

$x^2 + y^2 - 2x + 4y - 3 = 0$

Alternative

the centre of the reqd. circle is (1,-2)

the given circle is $x^2 + y^2 + 2y - 3 = 0$

then centre of this circle is$(0,-1)$

.:. The required circle is to pass through(0,-1)

.:. The radius of reqd.circle = distance between (1,-2)(0,-1)

$=> \sqrt{(1-0)^2 + (-2+1)^2}$

$=> \sqrt{(1+1)} = \sqrt{2}$

=> the equation of the rqd. circle is

haveing centre (1,-2) & r = $\sqrt{2}$

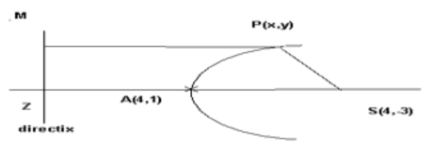
$(x-1)^2 + (y-(-2))^2 = (\sqrt{2})^2$

$=> x^2 + 1 - 2x + y^2 + 4 + 4y = 2$

$=> x^2 + y^2 - 2x + 4y + 3 = 0$

Q9 (b) Find the equation of the parabola whose focus is $(4,-3)$ and whose vertex is $(4,1)$.

Answer



Let A (4, 1) be the vertex and s (4,-3) be the focues let the axis meets the directrix at Z.
Let the Co-ordinates of Z be (x,y)
Since A is the mid pt of SZ. , We have

$$\frac{x_1+4}{2}=4, \frac{y_1-3}{2}=1$$

$$x_1+4=4, y_1-3=2$$

$$x_1=4, y_1=5$$

$$\therefore Z(4,5)$$

$$Slope\ of\ SZ=\frac{-3-5}{4-4}=\frac{-8}{0}(not\ \text{defined})$$

Also dixctrix is | to axis

$\therefore$ Slope of directrie = 0

equation of dixctrix passing through (4,5) and slope = 0

$$y-5=0(x-4)$$

$$y-5=0$$

let p(x, y) be any pt on the parabola draw PM | an the directries.

$\therefore$ PS = PM

$$\sqrt{(x-4)^2+(y+3)^2}=\left|\frac{y-5}{\sqrt{(1)^2+(0)^2}}\right|$$

Squareing both sides :-

$$x^2+16-8x+y^2+9+6y=y^2+25-10y$$

$$=> x^2-8x+16y=0$$

Text Books

1. Applied Mathematics for Polytechnics, H. K. Dass, 8th Edition, CBS Publishers & Distributors.

2. A Text book of Comprehensive Mathematics Class XI, Parmanand Gupta, Laxmi Publications (P) Ltd, New Delhi.

3. Engineering Mathematics, H. K. Dass, S, Chand and Company Ltd, 13th Edition, New Delhi.