Q2 (a) If $x^x y^y z^z = c$ **, show that at x=y=z,** $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Answer

Log of $x^x y^y z^z = c$, we get

$x \log x + y \log y + z \log z = \log c$................(1)

Diff both sides w.r.t x and y_1

$$x\frac{1}{x} + \log x + z\frac{1}{2}\frac{\partial z}{\partial x} + \log z\frac{\partial z}{\partial x} = 0 \text{ or } \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}..........(2)$$

$$y\frac{1}{y} + \log y + z\frac{1}{2}\frac{\partial z}{\partial y} + \log z\frac{\partial z}{\partial y} = 0 \text{ or } \frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log z}..........(3)$$

Diff (3) paste all w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log y}{(1+\log z)^2}.\frac{1}{z}\frac{\partial z}{\partial x}$$

$$= -\frac{1+\log y}{(1+\log z)^2}\frac{1}{z}\frac{1+\log x}{1+\log 2}(u \sin g(2))$$

$$At x = y = z$$

$$= -\frac{1+\log x}{(1+\log x)^2}\frac{1}{x}\frac{1+\log x}{1+\log x} = -\frac{1}{x(1+\log x)} = -[x(\log e + \log x)]^{-1}$$

$$= -[x \log ex]^{-1}$$

Q2 (b) Expand $f(x,y) = \tan^{-1}(xy)$ **in powers of** $(x-1)$ **and** $(y-1)$ **upto second degree terms.**

Answer

Here,

$$f(x,y)=\tan^{-1}(xy) \ldots\ldots\ldots \therefore\ f(1,1)=\tan^{-1}1=\frac{\pi}{4}$$

$$f_x(x,y)=\frac{y}{1+x^2y^2} \ldots\ldots\ldots \therefore\ f_2(1,1)=\frac{1}{2}$$

$$f_y(x,y)=\frac{x}{1+x^2y^2} \ldots\ldots\ldots\ f_y(1,1)=\frac{1}{2}$$

$$f_{xx}(x,y)=-\frac{-2y^3x}{(1+x^2y^2)^2} \ldots\ldots\ldots\ f_{xx}(1,1)=-\frac{1}{2}$$

$$f_{yy}(x,y)=-\frac{-2x^3y}{(1+x^2y^2)^2} \ldots\ldots\ldots\ f_{yy}(1,1)=-\frac{1}{2}$$

$$f_{xy}(x,y)=-\frac{1}{1+x^2y^2}-\frac{-2x^2y^2}{(1+x^2y^2)^2} \ldots\ldots\ldots\ f_{xy}(1,1)=\frac{1}{2}-\frac{1}{2}=0$$

$$\therefore\ f(x,y)=\tan^{-1}(xy)=\ f(1,1)+(x-1)f_x(1,1)+(y-1)f_y(1,1)+\frac{1}{\angle 2}(x-1)^2 f_{xx}(1,1)+\frac{1}{\angle 2}(y-1)^2$$

$$f_{yy}(1,1)+(x-1)(y-1)f_{xy}(1,1)+---------$$

$$=\frac{\pi}{4}+(x-1)\frac{1}{2}+(y-1)\frac{1}{2}+\frac{1}{\angle 2}(n-1)^2(-\frac{1}{2})+\frac{1}{\angle 2}(y-1)^2\left(-\frac{1}{2}\right)+(x-1)(y-1)....$$

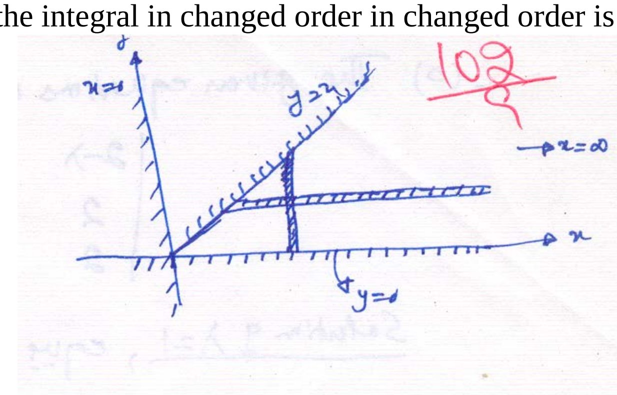
$$=\frac{\pi}{4}+\frac{1}{2}(x-1)+\frac{1}{2}(y-1)-\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2 e-----------$$

Q3 (a) Change the order of integration and then evaluate it $\int_0^{\infty}\int_0^{x} xe^{\frac{-x^2}{y}}\,dydx$

Answer

To change order of integration, we take strip parallel to x-axis to cover the each bounded by $x=0, x=\infty, y=0, y=x.$

Strip moves parallel to itself from y=0 to y=∞,keeping its ends on x=y two x=∞ hence the integral in changed order in changed order is



$$\int_{y=0}^{\infty}\int_{x=y}^{x=\infty} xe^{\frac{-x^2}{y}}\,dxdy$$

$$=\int_0^{\infty}\left[-\frac{y}{2}e^{-\frac{x^2}{y}}\right]_y^{\infty} dy$$

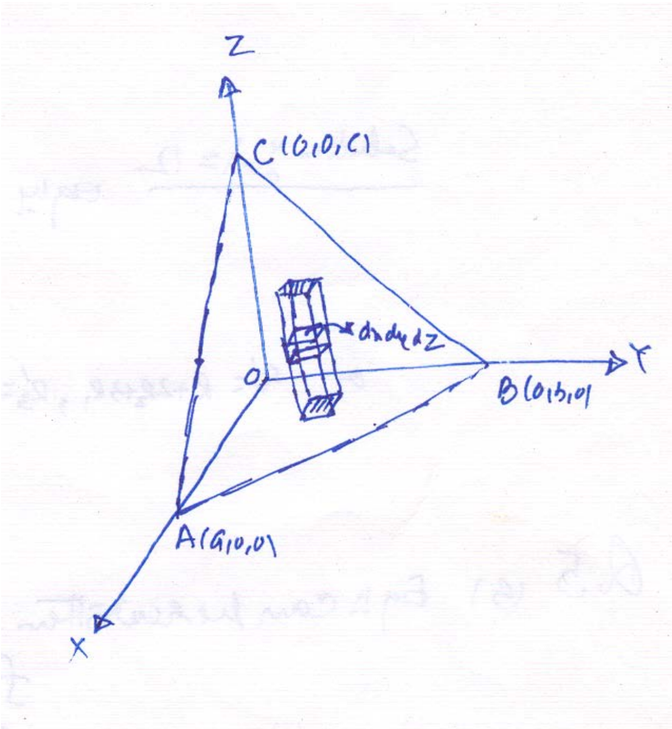
$$=\int_0^{\infty}-\frac{y}{2}\left(0-e^{-y}\right)dy=\frac{1}{2}\int_0^{\infty} ye^{-y}dy=\frac{1}{2}\left[-ye^{-y}\Big|_0^{\infty}+\int_0^{\infty}e^{-y}dy\right]$$

$$=\frac{1}{2}\left[-e^{-y}\right]_0^{\infty}=\frac{1}{2}$$

Q3 (b) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Answer

Volume of the tetrahedran bunneded by the coordinate to plnes and the plain



$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1n$$

$\iiint dz$ dy dx taken one entire val

$$\int_0^a\int_0^{b(1-n)}\int_0^z dz\,\text{dy dx}=\int_0^a\int_0^{b(1+y)} z\,\text{dy du}$$

$$=\int_0^a\int_0^{b(1-n/a)} c(1-\frac{x}{a}-\frac{y}{b})dy\,\text{dx}$$

$$=c\int_0^a\left[y-\frac{x}{a}y-\frac{1}{b}\frac{y^2}{2}\right]_0^{b(1-n/a)}dx=c\int_0^G\left[b\left(1-\frac{n}{a}\right)-\frac{nb}{a}\left(1-\frac{n}{a}1-\frac{b^2}{2b}\left(1-\frac{n}{a}\right)^2\right)\right]dx$$

$$=c\left[b\left(a-\frac{a^2}{2a}\right)-\frac{b}{a}\left(\frac{a^2}{2}-\frac{1}{a}\frac{a^3}{3}\right)-\frac{b}{2}\left(a-\frac{2}{a}\frac{a^2}{2}+\frac{1}{a^2}\frac{a^3}{3}\right)\right]$$

$$=c\left[ab-\frac{ab}{2}-\frac{ab}{2}+\frac{ab}{3}-\frac{ab}{2}+\frac{ab}{2}-\frac{ab}{6}\right]=\frac{abc}{6}$$

Q4 (a) Solve the equation $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix}=0$

Answer

0= $\begin{vmatrix} x+1 & 2x+1 & 3x+1 \\ 2x & 4x+3 & 6x+3 \\ 4x+1 & 6x+4 & 8x+4 \end{vmatrix}=\begin{vmatrix} x+1 & x & x \\ 2x & 2x+3 & 2x \\ 4x+1 & 2x+3 & 2x \end{vmatrix}=\begin{vmatrix} x+1 & x & x \\ -2 & 3 & 0 \\ 2x+1 & 0 & 0 \end{vmatrix}=-3x(2x+1)$

Hence $\text{x}=0$ or $\text{x}=-\frac{1}{2}$

Q4 (b) Find the values of λ **for which the equations** $(2-\lambda)\text{x}+2\text{y}+3=0,$ $2\text{x}+(4-\lambda)\text{y}+7=0,\ 2\text{x}+5\text{y}+(6-\lambda)=0$ **are consistent and find the values of x and y corresponding to each of these values of** λ**.**

Answer

The given equation will be consistent if

$$\begin{vmatrix} 2-\lambda & 2 & 3 \\ 2 & 4-\lambda & 7 \\ 2 & 5 & 6-\lambda \end{vmatrix} = 0 \text{ i.e}$$

$$\lambda^3 - 12\lambda^2 - \lambda + 12 = 0$$

i.e

$$\lambda = -1, = +1 = 12$$

soluting $g\lambda = 1$, equation become

$$x + 2y + 3 = 0$$

$$2x + 3y + 7 = 0$$

$$2x + 5y + 5 = 0$$

3[rd] equation is 4 times (1)-(2). :. 3[rd] n independent solution given by (1) and (2) and it is x=-5, y=1

Soluting y λ=-1, equation become

$$\begin{bmatrix} 3\,2 \\ 2\;\;5 \\ 2\;\;5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ +6 \end{bmatrix}$$

or

$$R_1^1 = R_1 + 2R_2 + 3R_1, R_3^1 = R_3 - R_2 \begin{bmatrix} 0 & 1 \\ 2 & -8 \\ 0 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 13 \end{bmatrix}$$

Hence $y = 1, x = \frac{1}{2}$

Q5 (a) Use Regula-Falsi method to compute the real root of $xe^x = 2$ correct to three decimal places.

Answer

Equation can be written as

$$f(x) = xe^x - 2 = 0.............(1)$$

$$f(0) = -2 \text{ and } f(1) = 0.718281828$$

$\therefore$ Root of (1) line between 0 and 1. By Regula falsi method , first apporch. is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-1(-2)}{2.718281828} = 0.73575888$$

$$f(x_1) = f(0.73575888) = -0.464423228$$

$\therefore$ *Root line between* 0.73575888 and 1.su second approch .root is given by

$$x_2 = \frac{0.73585888(.718281828) - 1(-.464423228)}{.718281828 + .464423228} = +0.83952077$$

$f(x_2) = f(+.83952077) = -0.0562935$

$\therefore$ *root* line between 0.82952077 and 1. so third approch . root is given by

Q5 (b) Use Runge-Kutta method of order four to find y(0.2) for the equation $\frac{dy}{dx} = \frac{y - x}{y + x}, y(0) = 1$. Take h = 0.2.

Answer

$$\frac{dy}{dx} = f(x, y) = \frac{y - x}{y + x}, x_0 = 0, y_0 = 1, h = 0.2$$

$$\therefore k_1 = hf(x_0, y_0) = 0.2x\frac{1 - 0}{1 + 0} = 0.2$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{ky}{2}) = 0.2\frac{(1 + .1) - (0 + .1)}{(1 + .1) + (0 + .1)} = \frac{0.2}{1.2} = 0.16666$$

Here $$k_3 = hf(x_0 + \frac{h}{2}, gy_0 + \frac{ky}{2}) = 0.2\frac{(1 + .08333) - (0 + .1)}{(1 + .08333) + (0 + .1)} = 0.16619$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\frac{(1 + .16619) - (0 + .2)}{(1 + .16619) + (0 + .2)} = 0.14144$$

$$\therefore k = \frac{k_1 + 2k_2 + 2k_3 + k_4)}{6} = 0.167836$$

Hence $y(0.2) = 1.167836$

Q6 (a) Solve the equation $\frac{dy}{dx} = -\left(\frac{x + y\cos x}{1 + \sin x}\right)$

Answer

Given equation can be written as

(x+y cos x)dx+(1+sin x)dy=0

Here M=x+y cos x, N=1+sin x

$$\frac{\partial M}{\partial y} = \cos x = \frac{\partial N}{\partial x}$$

$\therefore$ *Given* equ in each hence required solution is

$$\frac{x^2}{2} + y\sin x + y = k(a \text{ costant})$$

Q6 (b) Find the orthogonal trajectories of the family of coaxial circles $x^2 + y^2 + 2\lambda y + C = 0$**,** λ **being the parameter.**

Answer

Given family of coaxial circles is

$$x^2 + y^2 + 2\lambda y + C = 0.................(1)$$

$$diff\,(1), weget$$

$$2x + 2y\frac{dy}{dx} + 2\lambda\frac{dy}{dx} = 0.............(2)$$

$$\therefore Wronskian \text{ is given by}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3u} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

$$\therefore \text{P.I} = -y_1\int\frac{y_2 X}{W}dx + y_2\int\frac{y_1 X}{W}dx$$

$$= -e^{3x}\int\frac{xe^{3x}}{e^{6x}}\frac{e^{3x}}{x^2}dx + xe^{3x}\int\frac{e^{3x}}{e^{6x}}\frac{e^{3x}}{x^2}dx$$

$$= -e^{3x}\log x - e^{3x}$$

$$Hence \text{ compute solution is}$$

$$y = (c_1 + c_2 x)e^{3x} - e^{3x} - e^{3x}\log x$$

Q7 (a) Solve the differential equation $\frac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$

Answer

Q7 (b) Use method of variation of parameters to solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$

Answer

Q8 (a) Show that

(i) $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\int_0^{\pi/2}\frac{1}{\sqrt{\sin\theta}}d\theta = \pi$

(ii) $\beta(m, n+1) + \beta(m+1, n) = \beta(m, n)$

Answer

$$We\ know B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2x-1}\theta\, d\theta ....(1)$$

$$\therefore \int_0^{\pi/2} \sqrt{\sin\theta d\theta} = \frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{2}\right) by \text{ putting m - } \frac{3}{4}, x = \frac{1}{2} is....(1)$$

$$\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta = \frac{1}{2}\beta\left(\frac{1}{4},\frac{1}{2}\right) by \text{ putting m - } \frac{1}{4}, x = \frac{1}{2} is....(1)$$

multiplying we are

$$\int_0^{\pi/2}\sqrt{\sin\theta}\int_0^{\pi/2}\frac{1}{\sqrt{\sin\theta}}d\theta = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\frac{\sqrt{\frac{3}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{4}+\frac{1}{2}}}.\frac{\sqrt{\frac{1}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{4}+\frac{1}{2}}}$$

$$= \frac{1}{4}\frac{\sqrt{\frac{3}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{5}{4}}}.\frac{\sqrt{\frac{1}{4}}\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{4}}} = \frac{1}{4}\frac{\sqrt{\frac{1}{4}}}{\frac{1}{4}\sqrt{\frac{1}{4}}}\pi$$

$$U\sin g\sqrt{\frac{1}{2}} = \sqrt{\pi} = \pi$$

$$LHS = \beta(m,n+1)+\beta(m+1,n) = \frac{\sqrt{m}\sqrt{n+1}}{\sqrt{m+n+1}} + \frac{\sqrt{m+1}\sqrt{n}}{\sqrt{m+n+1}} = \frac{\sqrt{m}\sqrt[n]{n}+\sqrt[m]{m}\sqrt{n}}{\sqrt{m+n+1}}$$

$$= \frac{(m+n)\sqrt{m}\sqrt{n}}{(m+n)\sqrt{m+n}} = \frac{\sqrt{m}\sqrt{n}}{\big)m+n} = \beta(m,n) = RHS$$

Q8 (b) Solve in series the equation $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$

Answer

Here x=0 is an singular point because coefficient of $\frac{d^2y}{dx^2}$ become zero at x=0

$\therefore Let\ y = a_0 n^m + a_1 n^{m+1} + a_2 n^{m+2} + ...a_n n^{m+n} e...........$

$\therefore \frac{dy}{dx} = ma_0 n^{m-1} + a_1(m+1)n^m + a_2(m+2)(m+1)n^m + ... + a_n(m+n)(m+n-1)n^{m+n-2}.......$

$Sabstituting$ values is the given equation , we get

$ax(1-x)\left[a_0 m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + a_2(m+2)(m+1)x^m + ... + a_n(m+n)(m+n-1)x^{m+n-2}\right]$

$-12\left[a_0 mn^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + ... + a_x(m+n)n^{m+n-1} + ...\right]$

$+14\left[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + ... + a_n x^{m+n} + ...\right] = 0$

$Indicial$ equation is (equating to zex the coeff of low at powers of x

$9a_0 \ m(m-1) - 12a_0 m = 0$

$i.e$

$m = 0, \frac{7}{3} because\ a_0 \neq 0$

$\therefore Roots$ are distiner and do not differ . by an integer.

equation coeff of different powers of x, we get

$-9a_0 \ m(m-1) + 9a_1 \ (m+1)m - 12a_1(m+1) + 4a_0 = 0$

or

$3a_1 \ (m+1)(3m-4) = a_0 \ \left[9m^2 - 9m - 4\right]$

$= a_0(3m-4)(3m+1)$

$\therefore a_1 = \frac{3m+1}{3(m+1)} a_0$

$similarly$

$a_2 = \frac{3m+4}{3(m+2)} a_1 = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)} a_0$

$2\ fm = 0$

$a_1 = \frac{a_0}{3}, a_2 = \frac{4.1}{3^2.2.1} a_0, a_3 = \frac{7 \times 4 \times 1}{3^3 3.2.1} a_0$

$\therefore y_1 = a_0\left(1 + \frac{1}{3}x + \frac{1.4}{3^2 \angle 2}x^2 + \frac{1.4}{3^3 \angle 3}x^3 e.....\right)$

$\therefore y_2 = a_0 x^{\frac{7}{3}}\left(1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 e.....\right)$

$Hence$ compute solution is

$y = c_1 y_1 + c_2 y_2$

$= A_0\left(1 + \frac{1}{3}x + \frac{1.4}{3^2 \angle 2}x^2 + \frac{1.4.7}{3^3 \angle 3}x^3 e\right)$

$+ A_1 x^{\frac{7}{3}}\left(1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 e.....\right)$

Q9 (a) Show that $J_4(x)=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$

Answer
We know that

$$J_n(x)=\frac{x}{2x}[J_{n-1}|x|+J_{n+1}|n|]$$

$$\therefore J_{n+1}(x)=\frac{2x}{x}J_n|x|-J_{n-1}|n|$$

putting n = 1,2,3, we get

$$J_2(x)=\frac{2}{x}J_1,(x)-J_0(x)$$

$$J_3(x)=\frac{4}{x}J_2,(x)-J_1(x)$$

$$J_4(x)=\frac{6}{x}J_3,(x)-J_2(x)$$

$$=\frac{6}{x}\left[\frac{4}{x}J_2(x)-J_1(x)\right]-J_2(x)=$$

$$=\left(\frac{24}{x^2}-1\right)J_2(x)-\frac{6}{x}J_1(x)$$

$$=\left(\frac{24}{x^2}-1\right)\left(\frac{2}{x}J_1(x)-J_0(x)\right)-\frac{6}{x}J_1(x)$$

$$=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$$

Q9 (b) Show that $\int_{-1}^{1}\left(1-x^2\right)\ P'_m(x)\ \ P'_n(x)\,\mathbf{dx = 0}$

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Text Books

1. Higher Engineering Mathematics, Dr. B.S.Grewal, 40th edition 2007, Khanna publishers, Delhi.

2. Text book of Engineering Mathematics, N.P. Bali and Manish Goyal, 7th Edition 2007, Laxmi Publication (P) Ltd.