Mathematics SL

Internal assessment

First examinations 2006

Diploma Programme

Teacher support material

IBO

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International Baccalaureate Organization

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Mathematics SL: internal assessment—teacher support material

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General

The new course for mathematics SL will be examined for the first time in May 2006.

This publication supplements and should be read alongside the *Mathematics SL guide*, published in April 2004, which contains the curriculum and assessment requirements for the whole course. Where appropriate, extracts from the guide have been reproduced here for ease of reference. This publication offers useful suggestions and guidance for the implementation of the internally assessed component—the portfolio. General regulations and procedures relating to internal assessment have not been reproduced here but can be found in the relevant section of the *Vade Mecum*.

Every student must produce a portfolio containing **two** pieces of work completed during the course. Each piece of work in the portfolio is internally assessed by the teacher against criteria that are related to the objectives of the mathematics SL course. A sample of student portfolios from each school is then externally moderated to ensure uniformity of standards. The portfolio is worth 20% of the total score for the mathematics SL course.

Each task in a portfolio is assigned by the teacher. The tasks must be based on different areas of the course and represent **two** types of tasks: mathematical investigation (type I) and mathematical modelling (type II). The definitions of the different types of tasks are given on pages 2 and 3. Students must submit a portfolio containing **two** pieces of work, and it is recommended that teachers set more than two tasks.

This publication contains support material that exemplifies standards of assessment in the portfolio together with specimen tasks and student work, as well as guidance on the introduction and management of portfolio work in the classroom. This publication contains material contributed by teachers to help other teachers, across all aspects of portfolio work. Teachers are reminded that tasks that were written for the previous course do not lend themselves to achieving the highest levels in some criteria.

The IBO welcomes comments from teachers on this publication and also any ideas for teaching the course that may be of value to other teachers in Diploma Programme schools. Comments should be addressed to the group 5 subject area manager at the IBO.

The purpose of the portfolio

The purpose of the portfolio is to provide students with opportunities to be rewarded for mathematics carried out under ordinary conditions, that is, without the time limitations and pressure associated with written examinations. Consequently, the emphasis should be on good mathematical writing and thoughtful reflection.

The portfolio is also intended to provide students with opportunities to increase their understanding of mathematical concepts and processes. It is hoped that, by doing portfolio work, students benefit from these mathematical activities and find them both stimulating and rewarding.

The specific purposes of portfolio work are to:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete extended pieces of mathematical work without the time constraints of an examination

- enable students to develop individual skills and techniques, and to allow them to experience the satisfaction of applying mathematical processes on their own
- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- provide students with the opportunity to discover, use and appreciate the power of a calculator or computer as a tool for doing mathematics
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of the results they obtain
- provide opportunities for students to show, with confidence, what they know and what they can do.

Objectives

The portfolio is internally assessed by the teacher and externally moderated by the IBO. Assessment criteria have been developed to relate to the objectives of the mathematics SL course. In developing these criteria, particular attention has been given to the objectives listed here, since these cannot be easily addressed by means of timed, written examinations.

Where appropriate in the portfolio, students are expected to:

- know and use appropriate notation and terminology
- organize and present information and data in tabular, graphical and/or diagrammatic forms
- recognize patterns and structures in a variety of situations, and make generalizations
- demonstrate an understanding of and the appropriate use of mathematical modelling
- recognize and demonstrate an understanding of the practical applications of mathematics
- use appropriate technological devices as mathematical tools.

Tasks

Type I—mathematical investigation

While many teachers incorporate a problem-solving approach into their classroom practice, students also should be given the opportunity formally to carry out investigative work. The mathematical investigation is intended to highlight that:

- the idea of investigation is fundamental to the study of mathematics
- investigation work often leads to an appreciation of how mathematics can be applied to solve problems in a broad range of fields
- the discovery aspect of investigation work deepens understanding and provides intrinsic motivation
- during the process of investigation, students acquire mathematical knowledge, problem-solving techniques, a knowledge of fundamental concepts and an increase in self-confidence.

All investigations develop from an initial problem, the starting point. The problem must be clearly stated and contain no ambiguity. In addition, the problem should:

- provide a challenge and the opportunity for creativity
- contain multi-solution paths, that is, contain the potential for students to choose different courses of action from a range of options.

Essential skills to be assessed

- Producing a strategy
- Generating data
- Recognizing patterns or structures
- Searching for further cases
- Forming a general statement
- Testing a general statement
- Justifying a general statement
- Appropriate use of technology

Type II—mathematical modelling

Problem solving usually elicits a process-oriented approach, whereas mathematical modelling requires an experimental approach. By considering different alternatives, students can use modelling to arrive at a specific conclusion, from which the problem can be solved. To focus on the actual process of modelling, the assessment should concentrate on the appropriateness of the model selected in relation to the given situation, and on a critical interpretation of the results of the model in the real-world situation chosen.

Mathematical modelling involves the following skills.

- Translating the real-world problem into mathematics
- Constructing a model
- Solving the problem
- Interpreting the solution in the real-world situation (that is, by the modification or amplification of the problem)
- Recognizing that different models may be used to solve the same problem
- Comparing different models
- Identifying ranges of validity of the models
- Identifying the possible limits of technology
- Manipulating data

Essential skills to be assessed

- Identifying the problem variables
- Constructing relationships between these variables
- Manipulating data relevant to the problem
- Estimating the values of parameters within the model that cannot be measured or calculated from the data
- Evaluating the usefulness of the model
- Communicating the entire process
- Appropriate use of technology

Introduction

Please see pages 2 and 3 for definitions of the type I (mathematical investigation) and type II (mathematical modelling) tasks. Each piece of work in the portfolio should be assessed against the following six criteria.

Type I—mathematical investigation	Type II—mathematical modelling
A: Use of notation and terminology	A: Use of notation and terminology
B: Communication	B: Communication
C: Mathematical process—searching for patterns	C: Mathematical process—developing a model
D: Results—generalization	D: Results—interpretation
E: Use of technology	E: Use of technology
F: Quality of work	F: Quality of work

Descriptions of the achievement levels for each of these six assessment criteria appear in the *Mathematics SL guide* and are reproduced in this section for ease of reference. Instructions for applying the criteria are also included in the guide. Please note that criteria C and D are different for each type of task. It is particularly important to note that each achievement level represents the **minimum** requirement for that level to be awarded.

The final mark

Each portfolio must contain two pieces of work (if more than two pieces have been completed the best two should be selected for submission). The final mark for each portfolio is obtained by adding the achievement levels for both tasks together to give a total out of 40. For example:

	A		B		C		D		E		F		
Type I	1		3		3		2		3		2		
Type II	2		3		4		2		2		2		
	3	+	6	+	7	+	4	+	5	+	4	=	29

The final mark is 29.

Incomplete portfolios

If only one piece of work is submitted, award zero for each criterion for the missing work. For example:

	A		B		C		D		E		F		
Type I	1		3		3		2		3		2		
Type II	0		0		0		0		0		0		
	1	+	3	+	3	+	2	+	3	+	2	=	14

The final mark is 14.

Non-compliant portfolios

If two pieces of work are submitted, but they do not represent a type I task, mathematical investigation, and a type II task, mathematical modelling (as defined on pages 2 and 3), mark both tasks. Apply a penalty of 10 marks to the final mark. For example:

	A		B		C		D		E		F		
Type I	1		3		3		2		3		2		
Type II	2		3		4		2		2		2		
	3	+	6	+	7	+	4	+	5	+	4	=	29

Apply the penalty of 10 marks, so 29 – 10 = 19.

The final mark is 19.

Level of tasks

Teachers should set tasks that are appropriate to the level of the course. In particular, tasks appropriate to a standard level course, rather than to a higher level course, should be set.

Achievement levels

Criterion A: Use of notation and terminology

Achievement level	Descriptor
0	The student does **not** use appropriate notation and terminology.
1	The student uses **some** appropriate notation and/or terminology.
2	The student uses appropriate notation and terminology in a **consistent** manner and does so throughout the work.

Tasks will probably be set before students are aware of the notation and/or terminology to be used. Therefore the key idea behind this criterion is to assess how well students' use of terminology describes the context. Teachers should provide an appropriate level of background knowledge in the form of notes given to students at the time the task is set.

Correct mathematical notation is required, but it can be accompanied by calculator notation, particularly when students are substantiating their use of technology.

This criterion addresses appropriate use of mathematical symbols (for example, use of " $\approx$ " instead of " $=$ " and proper vector notation).

Word processing a document does not increase the level of achievement for this criterion or for criterion B.

Students should take care to write in appropriate mathematical symbols if the word-processing software does not supply them. For example, using $x \wedge 2$ instead of x^2 would be considered a lack of proper usage and the student would not achieve level 2.

Criterion B: Communication

Achievement level	Descriptor
0	The student **neither provides** explanations **nor uses** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).
1	The student **attempts** to provide explanations or uses **some** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).
2	The student provides **adequate** explanations or arguments, and communicates them using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).
3	The student provides **complete, coherent** explanations or arguments, and communicates them clearly using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).

This criterion also assesses how coherent the work is. The work can achieve a good mark if the reader does not need to refer to the wording used to set the task. In other words, the task can be marked independently.

Level 2 cannot be achieved if the student only writes down mathematical computations without explanation.

Graphs, tables and diagrams should accompany the work in the appropriate place and not be attached to the end of the document. Graphs must be correctly labelled and must be neatly drawn on graph paper. Graphs generated by a computer program or a calculator "screen dump" are acceptable provided that all items are correctly labelled, even if the labels are written in by hand. Colour keying the graphs can increase clarity of communication.

Criterion C: Mathematical process

Type I—mathematical investigation: searching for patterns

Achievement level	Descriptor
0	The student **does not attempt** to use a mathematical strategy.
1	The student **uses** a mathematical strategy to produce data.
2	The student **organizes** the data generated.
3	The student **attempts to analyse** data to enable the formulation of a general statement.
4	The student **successfully analyses** the **correct** data to enable the formulation of a general statement.
5	The student **tests** the validity of the general statement by considering further examples.

Students can only achieve level 3 if the amount of data generated is sufficient to warrant an analysis.

Type II—mathematical modelling: developing a model

Achievement level	Descriptor
0	The student **does not define** variables, parameters or constraints of the task.
1	The student defines **some** variables, parameters or constraints of the task.
2	The student defines variables, parameters **and** constraints of the task **and** attempts to create a mathematical model.
3	The student **correctly analyses** variables, parameters and constraints of the task to enable the **formulation** of a mathematical model that is **relevant** to the task and consistent with the level of the course.
4	The student **considers** how well the model fits the data.
5	The student **applies** the model to other situations.

At achievement level 5, applying the model to other situations could include, for example, a change of parameter or more data.

Criterion D: Results

Type I—mathematical investigation: generalization

Achievement level	Descriptor
0	The student **does not produce** any general statement consistent with the patterns and/or structures generated.
1	The student **attempts** to produce a general statement that is consistent with the patterns and/or structures generated.
2	The student **correctly** produces a general statement that is consistent with the patterns and/or structures generated.
3	The student **expresses** the **correct** general statement in **appropriate mathematical terminology**.
4	The student **correctly states** the scope or limitations of the general statement.
5	The student gives a **correct** informal justification of the general statement.

A student who gives a correct formal proof of the general statement that does not take into account scope or limitations would achieve level 4.

Type II—mathematical modelling: interpretation

Achievement level	Descriptor
0	The student **has not arrived** at any results.
1	The student has arrived at **some** results.
2	The student **has not interpreted** the reasonableness of the results of the model in the **context of the task**.
3	The student has **attempted** to interpret the reasonableness of the results of the model in the **context of the task**, to the appropriate degree of accuracy.
4	The student has **correctly** interpreted the reasonableness of the results of the model in the context of the task, to the appropriate degree of accuracy.
5	The student has **correctly** and **critically** interpreted the reasonableness of the results of the model in the context of the task, **including** possible limitations and modifications of these results, to the appropriate degree of accuracy.

Criterion E: Use of technology

Achievement level	Descriptor
0	The student uses a calculator or computer for **only routine** calculations.
1	The student **attempts** to use a calculator or computer in a manner that could enhance the development of the task.
2	The student makes **limited** use of a calculator or computer in a manner that enhances the development of the task.
3	The student makes **full** and **resourceful** use of a calculator or computer in a manner that **significantly** enhances the development of the task.

The level of calculator or computer technology varies from school to school. Therefore teachers should state the level of the technology that is available to their students.

Using a computer and/or a graphic display calculator (GDC) to generate only graphs or tables may not significantly contribute to the development of the task.

Criterion F: Quality of work

Achievement level	Descriptor
0	The student has shown a **poor** quality of work.
1	The student has shown a **satisfactory** quality of work.
2	The student has shown an **outstanding** quality of work.

Students who satisfy all the requirements correctly achieve level 1. For a student to achieve level 2, work must show precision, insight and a sophisticated level of mathematical understanding.

Overview of assessment criteria for type I tasks

	Criterion A: Use of notation and terminology	Criterion B: Communication	Criterion C: Mathematical process—searching for patterns	Criterion D: Results—generalization	Criterion E: Use of technology	Criterion F: Quality of work
0	The student does **not** use appropriate notation and terminology.	The student **neither provides** explanations **nor uses** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **does not attempt** to use a mathematical strategy.	The student **does not produce** any general statement consistent with the patterns and/or structures generated.	The student uses a calculator or computer for **only routine** calculations.	The student has shown a **poor** quality of work.
1	The student uses **some** appropriate notation and/or terminology.	The student **attempts** to provide explanations or uses **some** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **uses** a mathematical strategy to produce data.	The student **attempts** to produce a general statement that is consistent with the patterns and/or structures generated.	The student **attempts** to use a calculator or computer in a manner that could enhance the development of the task.	The student has shown a **satisfactory** quality of work.
2	The student uses appropriate notation and terminology in a **consistent** manner and does so throughout the work.	The student provides **adequate** explanations or arguments, and communicates them using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **organizes** the data generated.	The student **correctly** produces a general statement that is consistent with the patterns and/or structures generated.	The student makes **limited** use of a calculator or computer in a manner that enhances the development of the task.	The student has shown an **outstanding** quality of work.
3		The student provides **complete, coherent** explanations or arguments, and communicates them clearly using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **attempts to analyse** data to enable the formulation of a general statement.	The student **expresses** the **correct** general statement in **appropriate mathematical terminology**.	The student makes **full** and **resourceful** use of a calculator or computer in a manner that **significantly** enhances the development of the task.	
4			The student **successfully analyses** the **correct** data to enable the formulation of a general statement.	The student **correctly states** the scope or limitations of the general statement.		
5			The student **tests** the validity of the general statement by considering further examples.	The student gives a **correct**, informal justification of the general statement.		

Overview of assessment criteria for type II tasks

	Criterion A: Use of notation and terminology	Criterion B: Communication	Criterion C: Mathematical process—developing a model	Criterion D: Results—interpretation	Criterion E: Use of technology	Criterion F: Quality of work
0	The student does **not** use appropriate notation and terminology.	The student **neither provides** explanations **nor uses** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **does not define** variables, parameters or constraints of the task.	The student **has not arrived** at any results.	The student uses a calculator or computer for **only routine** calculations.	The student has shown a **poor** quality of work.
1	The student uses **some** appropriate notation and/or terminology.	The student **attempts** to provide explanations or uses **some** appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student defines **some** variables, parameters or constraints of the task.	The student has arrived at **some** results.	The student **attempts** to use a calculator or computer in a manner that could enhance the development of the task.	The student has shown a **satisfactory** quality of work.
2	The student uses appropriate notation and terminology in a **consistent** manner and does so throughout the work.	The student provides **adequate** explanations or arguments, and communicates them using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student defines variables, parameters **and** constraints of the task **and** attempts to create a model.	The student **has not interpreted** the reasonableness of the results of the model in the **context of the task**.	The student makes **limited** use of a calculator or computer in a manner that enhances the development of the task.	The student has shown an **outstanding** quality of work.
3		The student provides **complete, coherent** explanations or arguments, and communicates them clearly using appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).	The student **correctly analyses** variables, parameters and constraints of the task to enable the **formulation** of a mathematical model that is **relevant** to the task and consistent with the level of the course.	The student has **attempted** to interpret the reasonableness of the results of the model in the **context of the task**, to the appropriate degree of accuracy.	The student makes **full** and **resourceful** use of a calculator or computer in a manner that **significantly** enhances the development of the task.	
4			The student **considers** how well the model fits the data.	The student has **correctly** interpreted the reasonableness of the results of the model in the context of the task, to the appropriate degree of accuracy.		
5			The student **applies** the model to other situations.	The student has **correctly** and **critically** interpreted the reasonableness of the results of the model in the context of the task, **including** possible limitations and modifications of these results, to the appropriate degree of accuracy.		

Introducing students to portfolio work

This section deals with the issues that face the teacher when introducing portfolio work into the classroom. It deals specifically with the following:

- introducing students to portfolio work
- giving advice to students
- providing follow-up and feedback to students after the completion of each task
- ensuring that the portfolio work submitted is the student's own.

Using previous student work

Time in class could be used to allow students to mark student work from previous years (or student work from this document) against the assessment criteria.

The work produced could be exchanged between groups or individuals and marked by the students against the criteria. Alternatively, individual students or groups of students could mark their own work. This would allow students to get to know the criteria and understand their purpose.

Writing good mathematics

One of the main purposes of portfolio work is to help students to learn the importance of writing "good" mathematics. To understand the meaning of good mathematical writing it is suggested that students are encouraged from the start of the course to read published works in mathematics.

For example, students might be asked to read a suitable section of a mathematical textbook or an article from a mathematical journal in preparation for discussing it critically with the class. The emphasis here would not only be on understanding the mathematical content but also on developing an awareness of what is required for mathematical ideas to be communicated clearly to the reader.

Appropriate articles may be found in journals such as *Mathematics in Schools* (Mathematical Association, UK), *Mathematics Teacher* (National Council for Teachers of Mathematics, USA), *Explore* (International Schools Mathematics Teachers Foundation) and *Trigon* (Mathematical Association of South Australia).

Students should understand that the pieces of work they produce are for other people to read. The mathematics they write should therefore be clear and logical, and contain appropriate links and explanations.

Introductory tasks

Once the need to write good mathematics has been established, an introduction to portfolio work might be made by giving short, introductory tasks to the class as a whole, to smaller groups of students or to individual students.

The introductory tasks could include a piece of guided work where class time is spent discussing each step. There could also be discussion of how to structure the work, with particular emphasis on the way it should be written up. The introductory tasks should be simple tasks based on familiar work, so that the emphasis is on the process of completing a task.

Suitable topics for an introductory task might be fitting a function to a set of data (fitting a quadratic function to data by use of 3×3 simultaneous equations) or establishing general terms in sequences. Use of a regression tool to find a curve of best fit will enhance portfolio modelling tasks. Whichever topic is chosen, it should be integrated into the course of study by being directly relevant to the work that the students are currently being taught, rather than remaining an isolated piece of mathematics.

Giving advice to students

After students have read through a task, they may have questions. They should be encouraged to seek answers to these questions by referring to examples of their own work or to any available textbooks and/or by discussing particular issues with their teacher or with other students. Students are not expected to work in complete isolation on a task. In particular, teachers should not try to reproduce examination conditions.

Some students may need extra encouragement to overcome initial difficulties and misunderstandings, and teachers need not feel inhibited in giving advice to students. If students ask specific questions, teachers should, where appropriate, guide them into productive routes of inquiry rather than provide a direct answer. If students do the mathematics themselves and write up their own findings unaided then the work can be considered to be the students' own.

Providing follow-up and feedback

Most tasks will require follow-up work to ensure that students have understood the purpose of them, especially when the task is being used to introduce a topic or to consolidate understanding. Time spent on this counts as normal class teaching time.

It is also necessary to provide feedback to students on the individual achievement levels awarded for each criterion so that they can take action to improve their future performance. It is therefore important that students are provided with copies of the assessment criteria and are informed of the way in which achievement levels are awarded. Suggested ways of providing feedback are described in the section on management of the portfolio in the *Mathematics SL guide*.

Ensuring that the work submitted is the student's own

Students need to be aware from the beginning that any portfolio work submitted for assessment must be entirely their own work. Certain safeguards will need to be in place to ensure that this is the case. Suggestions are included in the *Mathematics SL guide*, in the section on guidance and authenticity in the internal assessment details.

Students may have easy access to information contained on the Internet, CD-Roms and other such sources. Although it may be very tempting for them to use a "cut and paste" technique, it should be made clear that this practice is unacceptable, unless the author is acknowledged and details of the source are provided.

Students may also wish to consult experts outside the school. While they should not be discouraged from seeking to expand their knowledge, students should be made aware of the extent to which they can obtain help from outside sources. Generally speaking, talking through a task and discussing related mathematical concepts is acceptable. However, it is not acceptable for students to receive written information on specific areas of a task or to have any part of the work completed for them.

Management of the portfolio

This section deals with the management of portfolio work. It summarizes the requirements stated in the *Mathematics SL guide* and gives suggestions for planning, record keeping and storage of student work.

These suggestions have been made by practising teachers to assist in providing a suitable framework for ensuring that all the relevant information is accessible when required. However, individual teachers may develop their own systems.

Planning

Portfolio work should be integrated into the course of study so that it enhances student learning by introducing a topic, reinforcing mathematical meaning or taking the place of a revision exercise. Therefore each task needs to correspond to the course of study devised by the individual teacher in terms of the knowledge and skills that the students have been taught.

Integration of portfolio work will occur more naturally if teachers write their own tasks. If a task written by someone else (for example, a colleague, a teacher from another school, or the IBO) is used, then adjustments may be needed to ensure that the mathematics is relevant to the topic being taught and to the background of the students.

Furthermore, it is important that each task:

- is comprehensive in terms of the information needed by students to complete the task
- is given a title, for ease of reference
- includes a reference to the type of task it is (that is, type I or type II)
- is printed in sufficient numbers that each student can have their own copy for reference.

Record keeping

It is important that well-documented records are kept throughout the two-year course so that teachers have the relevant information to complete the forms that accompany sample work submitted for moderation. Information about the requirements for submitting a sample for moderation, together with the necessary forms, can be found in the relevant section of the *Vade Mecum*.

It will be useful to record the following details at the time each task is set and assessed:

- the areas of the syllabus on which the task is based
- the date the task was given to the student and the date of submission (which should be approximately three to ten days later)
- the type of task (type I or type II)
- marking key (for example, numerical and/or algebraic results expected, particular points looked for in each criterion)
- the background to the task in relation to the skills and concepts from the syllabus which had, or had not, been taught to the students at the time the task was set
- the availability of technology.

This information is needed so that moderators can be aware of the context in which a task was set. For example, the expectations of performance in a task used to introduce a topic would be different from those in a similar task undertaken by students who had previously been taught the relevant mathematics.

Forms, similar to Form A at the end of this section, could be used to record these details.

Feedback to students

As stated in the section on giving guidance to students and in the *Mathematics SL guide*, it is important that students receive feedback on their own work so that they can improve their performance.

Form B at the end of this section has been devised for this purpose, but teachers may wish to amend it to suit their needs. It allows teachers to comment on the work submitted in relation to the achievement levels awarded for each criterion, and to give guidance on how improvements might be made. The assessment criteria could be printed on the back of this form for ease of reference by the student. Before handing back these forms to students, teachers should make copies for their own use so that they have a record of the achievement levels awarded.

Feedback to students may also, of course, be provided through discussion with individuals, small groups or through whole-class discussion. Teachers are encouraged to write feedback on the student work submitted.

Information for moderators and students

When marking student work, teachers are encouraged to note marks on work (such as ticks and crosses) and to include comments to explain why different achievement levels were awarded.

The moderator will need an explanation of how each achievement level was awarded. The relevant form from the *Vade Mecum* should be completed and submitted with the sample. However, if teachers have used Form B (or similar) for writing comments, this may be attached to the form from the *Vade Mecum*.

Storage

Once follow-up work has been completed, it is strongly recommended that students return their work to the teacher for safe storage. This helps to ensure that portfolios are not incomplete (that is, they do not contain fewer than **two** pieces of work) as a result of lost work.

Selecting the work for inclusion in the portfolio

If students have completed more than two pieces of work (as is recommended) they will need to select their best two pieces of work for inclusion in the portfolio. Form C at the end of this section, or a similar form, can be used for this purpose, and could be completed by the student during the course as an overall record of achievement.

It is suggested that the final selection be made by the student with guidance from the teacher. This could take place during class time and should therefore be included when planning the use of the **10** hours set aside for portfolio work.

The final selection needs to be considered carefully to ensure that all requirements are met and maximum credit is awarded. In particular, it should be remembered that each portfolio must contain **two** pieces of work, one of each type (type I and type II).

Selecting the sample

Teachers should ensure that they understand the requirements for submitting a sample. Incomplete work should not be included—a portfolio containing fewer than **two** pieces of work should not be part of the sample. If the sample selected by IBIS includes an incomplete portfolio, another portfolio with the same (or similar) mark should be submitted as well as the incomplete one.

Where there are two or more teachers of a subject within a school, they should agree on standards before arriving at the final mark for each student. That is, internal standardization of marks should take place within the school.

Deadlines

The deadlines for submitting a sample to the moderator are given in the *Vade Mecum*. Teachers should also check with their Diploma Programme coordinator to determine whether or not additional internal deadlines have been set by the school.

Tasks developed by teachers

Introduction

As stated in the *Mathematics SL guide*, portfolio tasks must be integrated into the course of study. This course of study should be devised before the start of the course and suitable tasks identified that can be incorporated into it to support the learning process. Students need to submit **two** pieces of work, but it is a good idea for them to be allowed to complete more than two and choose the best ones.

When setting tasks, the background of the students and the purpose of each task should be considered, as well as the types of technology available to students. The tasks should be:

- presented to students at appropriate times, periodically over the two-year course
- meaningful and relevant to the topic being studied at the time of the task
- considered as part of normal classwork and homework, not as something extra.

It may be helpful to provide students with a timetable of tasks at an early stage to assist them in managing their time. The following section deals with the cycle of development from possible starting points to the writing of a task.

Starting points

The process of developing a task can start from a number of different points.

A task written by someone else

It will be necessary to work through the task first to check its suitability. Amendments will almost certainly be needed for the task to be incorporated into a particular course of study. This includes the tasks in this document.

A syllabus topic to be covered

Some syllabus topics are suited to particular types of task. For example, sequences and series invite investigative work using a GDC, and exponential functions can be applied in a modelling task.

Outside sources

A report in a newspaper or journal can often provide the starting point for a modelling task or an investigation. Such a report provides an ideal opportunity to apply mathematics to real-life contexts. These reports may not appear at appropriate times in the course, so starting points of this kind usually require long-term planning.

Interesting points that arise in class discussion

Sometimes an interesting mathematics problem is exchanged among colleagues or arises from class discussion. If it is relevant to the syllabus it could be developed into a portfolio task.

Questions to ask before starting

The following questions need to be considered before starting to develop a portfolio task.

What is the purpose of the task?

The purpose of each task should be clearly understood in terms of whether it is being used to introduce a topic, reinforce mathematical meaning or take the place of a revision exercise.

What type should it be (type I or type II)?

It is important to make a decision about the type of task at an early stage and to make sure the task addresses the particular requirements of that type.

What part of the syllabus does it assess?

Portfolio tasks must relate directly to the syllabus. Choosing topics outside the syllabus, or extending work on topics beyond the intended level of study, will create extra work for students and teachers.

What knowledge and skills are involved?

Teachers should consider the prior knowledge and skills that are required for students to complete the task successfully. Teachers should also consider the mathematical knowledge and skills they wish students to obtain, develop and review as they work through the task.

What follow-up work will be needed?

The extent of the follow-up work required will vary with the nature of the task and should be planned in advance.

The cycle of development

In developing a portfolio task it will be necessary to work through a number of stages.

Stage 1

Draft the task, or select a task that has been written by someone else. The assessment criteria should be consulted at this point.

Stage 2

Work through the task yourself in full, as if you were a student.

Stage 3

Refer to the assessment criteria. Will the task provide an opportunity for students to gain the highest achievement levels?

Stage 4

Consider whether the task has achieved its aims. Is it of an appropriate length? Is it at an appropriate level? What will the students learn?

Stage 5

What flaws in the task have been exposed? How could the task be improved?

Stage 6

Redraft the task so that it will be ready to use with your students.

Stage 7

Present the task to your students, and then repeat stages 3 to 6.

Mathematics SL: The portfolio **Teacher's record**		**Form A**
Title of task:		**Type:** I II
Date set:	**Date submitted:**	
Syllabus topics covered		
Background information		
Purpose of the task		
Previous exposure to relevant concepts/skills		
Previous exposure to relevant terminology		
Available technology		
Teacher expectations regarding technology		

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name:		
Title of task:		**Type: I II**
Date set:	**Date submitted:**	
A Use of notation and terminology	/ 2	
B Communication	/ 3	
C Mathematical process	/ 5	
D Results	/ 5	
E Use of technology	/ 3	
F Quality of work	/ 2	

Mathematics SL: The portfolio **Form C**

Student record of achievement

Name:			Achievement levels awarded						Included in portfolio (✓)
No	**Title of task**	**Type**	**A (0–2)**	**B (0–3)**	**C (0–5)**	**D (0–5)**	**E (0–3)**	**F (0–2)**	
1									
2									
3									
4									
5									
6									
	Final mark for each achievement level →								

Overall final mark

Introduction

This section contains eight specimen tasks that have been developed by teachers. A list of titles follows, together with information on the type of each task and the areas of the course on which it is based. The list also gives an indication of whether examples of Form A and student work are available (see the section containing examples of student work and assessment).

Title of task		Type	Areas of the syllabus	Example of Form A included	Student work included
1	Continued fractions	I	1.1, 7.1		
2	The Koch snowflake	I	1.1, 7.1		
3	Matrix powers	I	4.2, 4.3	✓	✓
4	Derivatives of sine functions	I	7.1, 7.2	✓	
5	Stopping distances	II	2.2, 2.3, 2.5		✓
6	Sunrise over New York	II	2.2, 2.3, 3.4		
7	Tide modelling	II	2.2, 2.3, 3.4		
8	Modelling the amount of a drug in the bloodstream	II	2.2, 2.3, 2.8		✓

Each task is classified according to its type as follows.

- Type I: mathematical investigation
- Type II: mathematical modelling

If these tasks are used by teachers, it is important that they work through them carefully to ensure that the instructions are clear and that the material is appropriate for their students. (See also the section on tasks developed by teachers.) There is no prescribed style for the formatting of tasks. Those that appear in this publication have been formatted in a particular way, but there is no requirement for teachers to do the same.

Type I tasks

1 Continued fractions

SL Type I

Description

Consider the continued fraction below.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}}}$$

We can consider this "infinite fraction" as a sequence of terms, t_n, where

$$t_1 = 1 + 1$$

$$t_2 = 1+\frac{1}{1+1}$$

$$t_3 = 1+\cfrac{1}{1+\cfrac{1}{1+1}}$$

•

•

•

Method

1. Determine a generalized formula for t_{n+1} in terms of t_n.
2. Compute the decimal equivalents of the first 10 terms. Enter the terms into a data table and plot the relation between n and t_n using a GDC or computer. Provide printed output of your plot. What do you notice? What does this suggest about the value of $t_n - t_{n+1}$ as n gets very large?
3. What problems arise when you try to determine the 200th term?
4. Use the results of step 1 and step 2 to establish an **exact** value for the continued fraction.

5. Now consider another continued fraction.

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\cdots}}}}}}}$$

Repeat steps 1 to 4 using this continued fraction.

6. Now consider the general continued fraction.

$$k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{k+\cfrac{1}{\cdots}}}}}}}$$

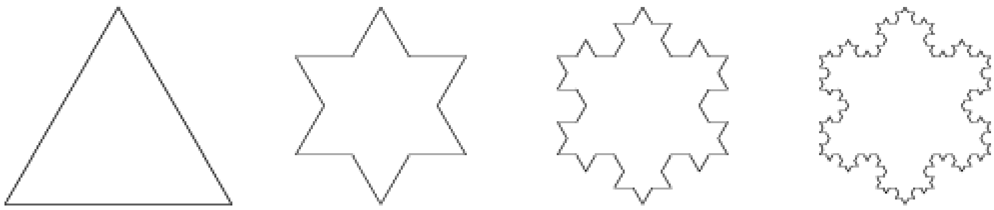
By considering other values of k, determine a generalized statement for the exact value of any such continued fraction. For which values of k does your generalized statement hold true? How do you know? Provide evidence to support your answer.

2 The Koch snowflake

SL Type I

Description

In 1904 Helge von Koch identified a fractal that appeared to model the snowflake. The fractal is built by starting with an equilateral triangle and removing the inner third of each side, building another equilateral triangle at the location where the side was removed, and then repeating the process indefinitely. The process is illustrated clearly below, showing the original triangle at stage 0 and the resulting figures after one, two and three iterations.



Stage 0 Stage 1 Stage 2 Stage 3

Method

Let N_n = the number of sides, l_n = the length of a single side, P_n = the length of the perimeter and A_n = the area of the snowflake, at the n^{th} stage.

1. Using an initial side length of 1, create a table that shows the values of N_n , l_n , P_n and A_n for $n = 0$, 1, 2 and 3. Use exact values in your results. Explain the relationship between successive terms in the table for each quantity N_n , l_n , P_n and A_n.
2. Using a GDC or a suitable graphing software package, create graphs of the four sets of values plotted against the value of n. Provide separate printed output for each graph.
3. For each of the graphs above, develop a statement in terms of n that generalizes the behaviour shown in its graph. Explain how you arrived at your generalizations. Verify that your generalizations apply consistently to the sets of values produced in the table.
4. Investigate what happens at $n = 4$. Use your conjectures from step 3 to obtain values for N_4 , l_4 , P_4 and A_4. Now draw a large diagram of one "side" (that is, one side of the original triangle that has been transformed) of the fractal at stage 4 and clearly verify your predictions.
5. Calculate values for N_6 , l_6 , P_6 and A_6. You need not verify these answers.
6. Write down successive values of A_n in terms of A_0. What pattern emerges?
7. Explain what happens to the perimeter and area as n gets very large. What conclusion can you make about the area as $n \to \infty$? Comment on your results.

3 Matrix powers

SL Type I

Method

1. Consider the matrix $\boldsymbol{M}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$.

   Calculate $\boldsymbol{M}^{n}$ for $n = 2, 3, 4, 5, 10, 20, 50$.

   Describe in words any pattern you observe.

   Use this pattern to find a general expression for the matrix $\boldsymbol{M}^{n}$ in terms of n.

2. Consider the matrices $\boldsymbol{P}=\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$ and $\boldsymbol{S}=\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}$.

   $$\boldsymbol{P}^{2}=\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}^{2}=\begin{pmatrix}10 & 6\\ 6 & 10\end{pmatrix}=2\begin{pmatrix}5 & 3\\ 3 & 5\end{pmatrix};\ \boldsymbol{S}^{2}=\begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}^{2}=\begin{pmatrix}20 & 16\\ 16 & 20\end{pmatrix}=2\begin{pmatrix}10 & 8\\ 8 & 10\end{pmatrix}$$

   Calculate $\boldsymbol{P}^{n}$ and $\boldsymbol{S}^{n}$ for other values of n and describe any pattern(s) you observe.

3. Now consider matrices of the form $\begin{pmatrix}k+1 & k-1\\ k-1 & k+1\end{pmatrix}$.

   Steps 1 and 2 contain examples of these matrices for $k = 1, 2$ and 3.

   Consider other values of k, and describe any pattern(s) you observe.

   Generalize these results in terms of k and n.

4. Use technology to investigate what happens with further values of k and n. State the scope or limitations of k and n.

5. Explain why your results hold true in general.

Mathematics SL: The portfolio **Form A**

Teacher's record

Title of task: Matrix powers	**Type:** I II
Date set: 12/03/05	**Date submitted:** 20/03/05

Syllabus topics covered

4.2, 4.3

Background information

Purpose of the task

To provide further practice and appreciation of powers of matrices. In considering scope and limitations I hoped that students would discover that only integer powers are defined.

Previous exposure to relevant concepts/skills

Students have multiplied matrices but not considered powers of matrices.

Previous exposure to relevant terminology

Students are familiar with relevant terminology except inverse of a matrix.

Available technology

Students each have a Texas Instruments -83 plus™, and Windows Graphlink® software is always available in the computer lab. Students also have access to Autograph® software.

Teacher expectations regarding technology

For this task I did not expect students to provide printout; handwritten matrices would be sufficient. I did expect that they would acknowledge where they made use of GDC or other technology. I was able to see use of calculators in initial and final classroom sessions on the task.

4 Derivatives of sine functions

SL Type I

Please note that this task is only suitable for use before students have studied derivatives of sine functions and the chain rule. It is a way of introducing the topics graphically.

Method

1. Investigate the derivative of the function $f(x)=\sin x$.
   - Graph the function $f(x)=\sin x$ for $-2\pi \le x \le 2\pi$.
   - Based only on this graph, describe as carefully and fully as you can, the behaviour of the gradient of the function on the given domain. Include a sketch of the graph of $y=f'(x)$.
   - Make a conjecture for the derived function.
   - Use your calculator to test your conjecture graphically. You may find the nDeriv function useful. Explain your method and your findings. Modify your conjecture if necessary.
   - Use your calculator to verify your conjecture numerically. Explain your method and your findings. Modify your conjecture if necessary.
2. Investigate the derivatives of functions of the form $g(x)=a\sin x$ in a similar way.
   - Consider several different values of a.
   - Make a conjecture for $g'(x)$.
   - Test the conjecture with further examples.
   - State for what values of a the conjecture holds.
3. Investigate the derivatives of functions of the form $h(x)=\sin bx$ in a similar way.
4. Investigate the derivatives of functions of the form $j(x)=\sin(x+c)$ in a similar way.
5. Use your results to steps 1 to 4 to make a conjecture for the derivative of $k(x)=a\sin b(x+c)$. Choose a value for a, for b and for c. Verify your conjecture for this particular case.
6. Consider $m(x)=\sin^2 x$. Investigate the derivative of this function as before and show that it can be written as $m'(x)=2\sin x\cos x$.

Mathematics SL: The portfolio **Form A**

Teacher's record

Title of task: Derivatives of sine functions	**Type:** I II
Date set: 12/03/05	**Date submitted:** 20/03/05

Syllabus topics covered

7.1, 7.2

Background information

Purpose of the task

To introduce and make plausible the derivatives of sin *x* and simple composites with sin *x*.

Previous exposure to relevant concepts/skills

Students are familiar with the derivative as a function to describe gradient, with transformations of graphs of trig functions and with some double angle formulae.

Previous exposure to relevant terminology

Students are familiar with all relevant terminology.

Available technology

Students each have a Texas Instruments -83 plus™, and Windows Graphlink® software is always available in the computer lab. Students also have access to Autograph® software.

Teacher expectations regarding technology

Students have previously used nDeriv to graph the gradient function and are competent in use of tables. These are the approaches expected.

Calculator screen dumps or graphs printed out from Autograph® are expected in support of the work done.

5 Stopping distances

SL Type II

Description

When a driver stops her car, she must first think to apply the brakes. Then the brakes must actually stop the vehicle.

The table below lists average times for these processes at various speeds.

Speed (kmh^{-1})	Thinking distance (m)	Braking distance (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

In this task you will develop individual functions that model the relationships between speed and thinking distance, as well as speed and braking distance. You will also develop a model for the relationship between speed and overall stopping distance.

Method

1. Use a GDC or graphing software to create two data plots: speed versus thinking distance and speed versus braking distance. Describe your results.
2. Using your knowledge of functions, develop functions that model the behaviours noted in step 1. Explain your work.
3. The overall stopping distance is obtained from adding the thinking distance to the braking distance. Create a data table of speed and overall stopping distance. Graph this data and describe the results.
4. Develop a function that models the relationship between speed and overall stopping distance. How is this function related to the functions obtained in step 2?
5. Overall stopping distances for other speeds are given below. Discuss how your model fits this data, and what modifications might be necessary.

Speed (kmh^{-1})	Stopping distance (m)
10	2.5
40	17
90	65
160	180

6 Sunrise over New York

SL Type II

Description

The table shows times of sunrise over New York at weekly intervals during 2003, starting on 1 January. All times are Eastern Standard Time in hours and minutes.

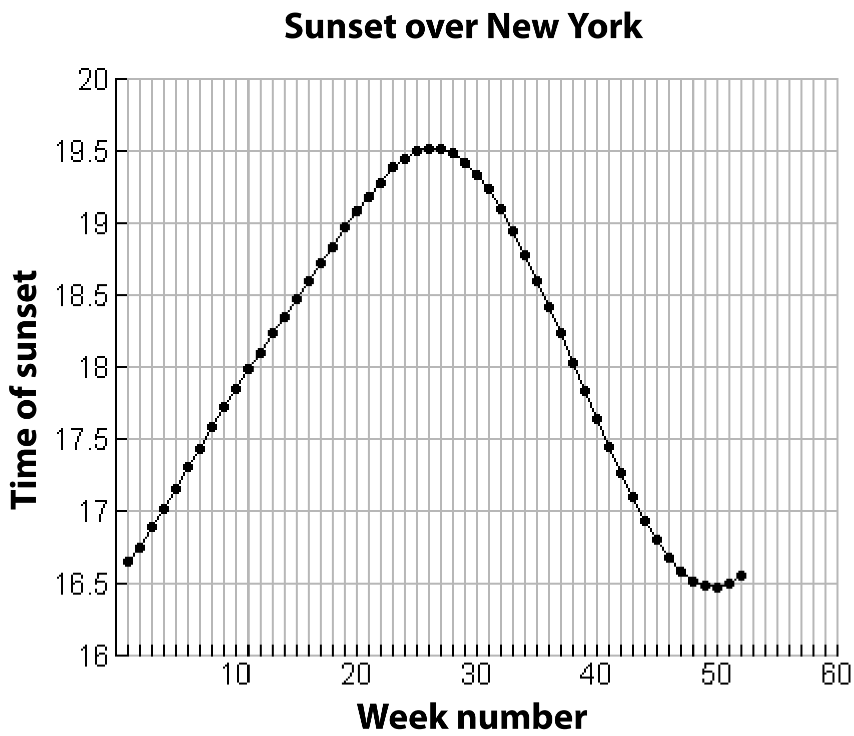
Time	Week	Time	Week	Time	Week
07.20	1	04.34	21	06.06	41
07.20	2	04.29	22	06.14	42
07.18	3	04.26	23	06.22	43
07.14	4	04.24	24	06.30	44
07.09	5	04.26	25	06.39	45
07.02	6	04.29	26	06.47	46
06.54	7	04.33	27	06.55	47
06.45	8	04.38	28	07.02	48
06.35	9	04.44	29	07.08	49
06.24	10	04.50	30	07.14	50
06.13	11	04.57	31	07.18	51
06.01	12	05.04	32	07.18	52
05.50	13	05.11	33		
05.38	14	05.17	34		
05.27	15	05.24	35		
05.16	16	05.31	36		
05.06	17	05.38	37		
04.56	18	05.45	38		
04.47	19	05.52	39		
04.40	20	05.59	40		

Source: http://www.aa.usno.navy.mil/

Method

1. Use a graphing package or spreadsheet to draw a graph of this data.
2. What type of function might be suitable to model this data? Explain what assumptions you are making.
3. Use your knowledge of the graphs of such functions to find a suitable function that models the behaviour. Identify any variables and parameters clearly, and explain how you determined them. Comment on how well your function fits the data.
4. Use a regression tool to find the best fit function. Comment on any differences from the function you found in step 3.
5. Compare the usefulness of the two models to:
   (a) someone planning a daybreak run
   (b) someone programming switching off the street-lighting.
6. How would each of these change if you travelled 1000km:
   (a) north
   (b) south
   (c) west?

   Use the Internet to find data to support your answers.
7. This graph shows the corresponding times of sunset.



   Use this information together with one of the models for sunrise to find estimates for:
   (a) the length of the shortest day
   (b) the approximate dates between which the day is more than 12 hours long.

7 Tide modelling

SL Type II

Description

The Bay of Fundy in Nova Scotia, Canada is deemed to have the greatest average change in tide height in the world. In the table below data is presented from 27 December 2003 using Atlantic Standard Time (AST). The heights were taken at Grindstone Island.

Time (AST)	00.00	01.00	02.00	03.00	04.00	05.00	06.00	07.00	08.00	09.00	10.00	11.00
Height (m)	7.5	10.2	11.8	12.0	10.9	8.9	6.3	3.6	1.6	0.9	1.8	4.0

Time (AST)	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00	21.00	22.00	23.00
Height (m)	6.9	9.7	11.6	12.3	11.6	9.9	7.3	4.5	2.1	0.7	0.8	2.4

Source: http://www.lau.chs-shc.dfo-mpo.gc.ca

In this task you will develop a model function for the relationship between time of day and the height of the tide. Consider carefully the expectations of a modelling task as you complete your work.

Method

1. Using a GDC or graphing software, plot the graph of time against height. Describe the result.
2. Use your knowledge of functions to develop a function that models the behaviour noted in the graph. Describe any variables, parameters or constraints for the model. Explain clearly how you established the value of any parameters.
3. Draw a graph of your function on the same set of axes as the graph in step 1. How well does the function fit the data?
4. Modify the function to create a better fit. Describe the issues you had to consider.
5. Good sailors will launch their boats on an outgoing tide (that is when the tide is going out). Use your model to determine the times between which a good sailor would have launched a boat on 27 December 2003.
6. Use the regression feature of your GDC or software to develop a best-fit function for this data. Compare this function with the one you developed analytically.
7. The table below lists the tide heights for 28 December 2003. Does your function fit these data? What modifications are needed? Confirm that your modified model fits the data.

Time (AST)	00.00	01.00	02.00	03.00	04.00	05.00	06.00	07.00	08.00	09.00	10.00	11.00
Height (m)	5.0	7.9	10.2	11.6	11.6	10.5	8.5	6.0	3.5	1.7	1.2	2.2

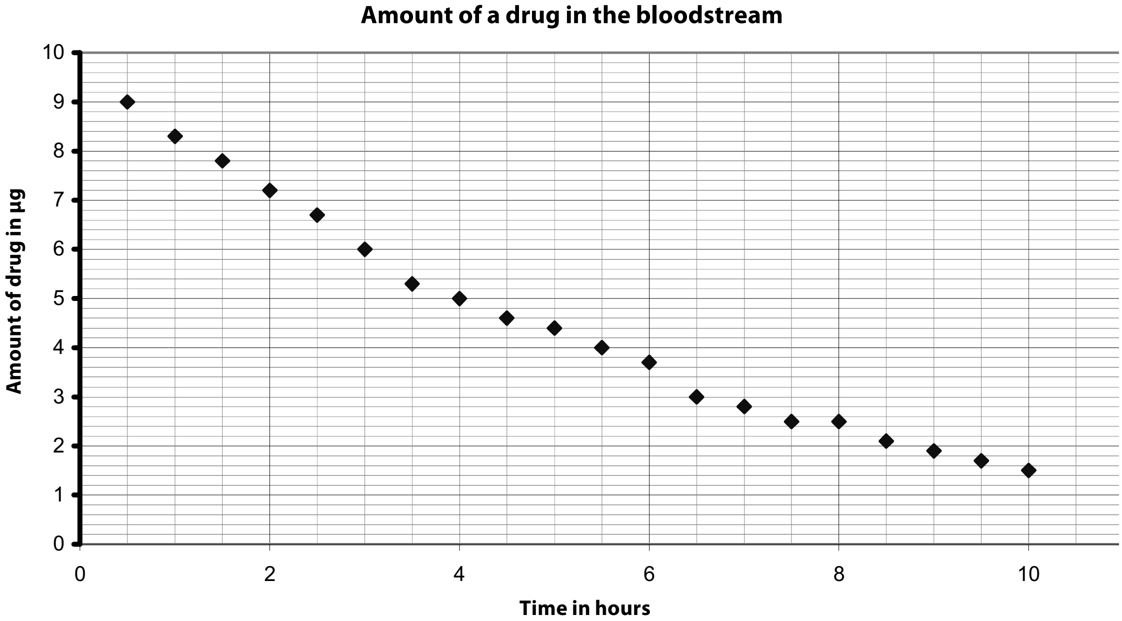
Time (AST)	12.00	13.00	14.00	15.00	16.00	17.00	18.00	19.00	20.00	21.00	22.00	23.00
Height (m)	4.4	7.2	9.7	11.3	11.8	11.1	9.4	7.0	4.4	2.2	1.0	1.3

8 Modelling the amount of a drug in the bloodstream

SL Type II

Description

The graph below records the amount of a drug for treating malaria in the bloodstream over the 10 hours following an initial dose of 10µg.



It seems that the rate of decrease of the drug is approximately proportional to the amount remaining.

Method

Part A

1. Use this information to help you find a suitable function to model this data.
2. Draw a graph of your function and compare your graph to the one above.
3. Comment on the suitability of the model.

Part B

A patient is instructed to take 10µg of this drug every six hours.

1. **Sketch** a diagram to show the amount of the drug in the bloodstream over a 24-hour period and state any assumptions made.
2. Use your GDC or graphing software and your model from part A to draw an accurate graph to represent this situation.
3. State the maximum and minimum amounts during this period.
4. Describe what would happen to these values over the next week if:
   (a) no further doses are taken
   (b) doses continue to be taken every six hours.

Introduction

This section contains examples of student work that have been produced from the tasks in the previous section.

Student	Title of task	Number of specimen portfolio task
A	Matrix powers	3
B	Matrix powers	3
C	Stopping distances	5
D	Modelling the amount of a drug in the bloodstream	8

The numbers of these tasks, listed in the third column, refer to the order in which they appear in the section entitled "Specimen portfolio tasks".

The assessment for each piece of work appears at the end of the work. The intention is to demonstrate the overall standards required for mathematics SL and to illustrate how the achievement levels for each criterion should be awarded.

The assessment was undertaken by a team of senior examiners and teachers. Each piece of work has been marked more than once and agreement reached on the achievement level to be awarded. Explanations are given for the award of each achievement level and, where appropriate, specific references to the work of the student have been made.

Teachers may wish to mark the student work themselves before reading the assessment to compare their standard of marking against that of the examiners.

Matrix powers—student A

SL Type I

Matrix Powers

Lacks an introduction

1. $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$M^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+0 \\ 0+0 & 0+4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

Using GDC;

$M^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, $M^4 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

$M^5 = \begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix}$, $M^{10} = \begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix}$

$$M^{20} = \begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix}$$

$$M^{50} \doteq \begin{pmatrix} 1.13 \times 10^{15} & 0 \\ 0 & 1.13 \times 10^{15} \end{pmatrix}$$

For each higher power of M the entries double.

So $M^2 = 2 \cdot M$

$M^3 = 2 \cdot M^2 = 2^2 \cdot M$

$M^4 = 2 \cdot M^3 = 2^3 \cdot M$

$\vdots$

$M^n = 2^{n-1} M$

– 1 –

2. $P = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

just enough data

$P^2 = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

by GDC $P^3 = \begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix} = 2 \begin{pmatrix} 18 & 14 \\ 14 & 18 \end{pmatrix} = 2^2 \begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}$

$P^4 = \begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix} = 2 \begin{pmatrix} 68 & 60 \\ 60 & 68 \end{pmatrix}$

$= 2^2 \begin{pmatrix} 34 & 30 \\ 30 & 34 \end{pmatrix} = 2^3 \begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix}$

I will continue taking powers of 2 out.

$P^5 = \begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix} = 2^4 \begin{pmatrix} 33 & 31 \\ 31 & 33 \end{pmatrix}$ ✓

It seems that P^n will have a factor 2^{n-1}. If we look at Part 4, this gives a hint for the structure of the remaining matricies.

i.e. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2+1 & 2-1 \\ 2-1 & 2+1 \end{pmatrix}$

$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2^2+1 & 2^2-1 \\ 2^2-1 & 2^2+1 \end{pmatrix}$

just enough data

$\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix} = \begin{pmatrix} 2^3+1 & 2^3-1 \\ 2^3-1 & 2^3+1 \end{pmatrix}$

– 2 –

2. cont'd

$$\begin{pmatrix} 17 & 15 \\ 15 & 17 \end{pmatrix} = \begin{pmatrix} 2^4+1 & 2^4-1 \\ 2^4-1 & 2^4+1 \end{pmatrix}$$

Therefore it appears that the pattern is;

$$P^n = 2^{n-1}\begin{pmatrix} 2^n+1 & 2^n-1 \\ 2^n-1 & 2^n+1 \end{pmatrix}$$

Check!

$P^{10} =$ (by GDC) $\begin{pmatrix} 524800 & 523776 \\ 523776 & 524800 \end{pmatrix}$

$$= 512\begin{pmatrix} 1025 & 1023 \\ 1023 & 1025 \end{pmatrix}$$

$$= 2^9\begin{pmatrix} 2^{10}+1 & 2^{10}-1 \\ 2^{10}-1 & 2^{10}+1 \end{pmatrix}$$

For $S = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$

(by GDC)

$$S^2 = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 2\begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

not enough data generated

$$S^3 = \begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix} = 4\begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix}$$

This looks different, as we could take out another factor of 2.

– 3 –

2. Cont'd

However, $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3+1 & 3-1 \\ 3-1 & 3+1 \end{pmatrix}$

So in $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$ $k = 3$

Now $S^2 = 2 \cdot \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3^2+1 & 3^2-1 \\ 3^2-1 & 3^2+1 \end{pmatrix}$

$S^3 = 4 \begin{pmatrix} 28 & 26 \\ 26 & 28 \end{pmatrix} = 2^2 \begin{pmatrix} 3^3+1 & 3^3-1 \\ 3^3-1 & 3^3+1 \end{pmatrix}$

So it appears that the pattern might be;

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$

Check!

$S^5 = \begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$ (by GDC)

$= 16 \begin{pmatrix} 244 & 242 \\ 242 & 244 \end{pmatrix}$

$= 2^4 \begin{pmatrix} 3^5+1 & 3^5-1 \\ 3^5-1 & 3^5+1 \end{pmatrix}$

So, in general

$$S^n = 2^{n-1} \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$ correct

– 4 –

3. For $M = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

we expect

$$M^n = 2^{n-1}\begin{pmatrix} k^n+1 & k^n-1 \\ k^n-1 & k^n+1 \end{pmatrix}$$

4. Using the GDC I will try some other values of k and n.

Only one further example checked
No discussion of scope or limitations

$k=5$, $n=7$

$$M^7 = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7$$

by the pattern, this should be;

$$M^7 = 2^6\begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

$2^6 = 64$ $\quad 5^7+1 = 78125+1 = 78126$

$5^7-1 = 78125-1 = 78124$

by GDC;

$$\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}^7 = \begin{pmatrix} 5000064 & 4999936 \\ 4999936 & 5000064 \end{pmatrix}$$

$$= 64\begin{pmatrix} 78126 & 78124 \\ 78124 & 78126 \end{pmatrix}$$

$$= 2^6\begin{pmatrix} 5^7+1 & 5^7-1 \\ 5^7-1 & 5^7+1 \end{pmatrix}$$

- 5 -

5. Consider $M = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$

$$M^2 = \begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)^2 + (k-1)^2 & (k+1)(k-1) + (k-1)(k+1) \\ (k-1)(k+1) + (k+1)(k-1) & (k-1)^2 + (k+1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} k^2+2k+1+k^2-2k+1 & k^2-1+k^2-1 \\ k^2-1+k^2-1 & k^2+2k+1+k^2-2k+1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k^2+2 & 2k^2-2 \\ 2k^2-2 & 2k^2+2 \end{pmatrix}$$

$$= 2\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}$$, which fits the pattern.

Now

$$M^3 = M^2 \cdot M = 2\begin{pmatrix} k^2+1 & k^2-1 \\ k^2-1 & k^2+1 \end{pmatrix}\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 2k^3+2 & 2k^3-2 \\ 2k^3-2 & 2k^3+2 \end{pmatrix}$$

$$= 2^2\begin{pmatrix} k^3+1 & k^3-1 \\ k^3-1 & k^3+1 \end{pmatrix}$$

So, the pattern works !!

This page provides a suitable informal justification

- 6 -

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student A		
Title of task: Matrix powers		**Type:** I II
Date set: 12/03/05	**Date submitted:** 20/03/05	
A Use of notation and terminology	2 / 2	
B Communication	2 / 3	
No introduction, more explanation needed. Hard to follow in places.		
C Mathematical process	4 / 5	
Successful analysis. Only just enough data to warrant level above 2. Only one further example looked at, not enough for 5.		
D Results	3 / 5	
A lack of any discussion of scope or limitations prevents the award of a higher level, even though an informal justification is given.		
E Use of technology	2 / 3	
Not fully exploited. The GDC allows for fuller explanation of other cases.		
F Quality of work	1 / 2	
Satisfactory		

Matrix powers—student B

SL Type I

1. This assignment is looking for a general formula that will give the nth power of the matrix $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}$.

good, B concise introduction!

The simplest matrix of this form is

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This table gives results for various values of n

n	M^n
1	$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^2 \end{pmatrix}$
3	$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 2^3 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 2^4 & 0 \\ 0 & 2^4 \end{pmatrix}$
5	$\begin{pmatrix} 32 & 0 \\ 0 & 32 \end{pmatrix} = \begin{pmatrix} 2^5 & 0 \\ 0 & 2^5 \end{pmatrix}$
10	$\begin{pmatrix} 1024 & 0 \\ 0 & 1024 \end{pmatrix} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$
20	$\begin{pmatrix} 1048576 & 0 \\ 0 & 1048576 \end{pmatrix} = \begin{pmatrix} 2^{20} & 0 \\ 0 & 2^{20} \end{pmatrix}$

how? GDC? Say so! E

The elements of the leading diagonal are powers of 2. The other two elements are zero in every case

$$M^n = \begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$$

✓

2. The table gives results for $P^n = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^n$ and $S^n = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}^n$

n	P^n	S^n
1	$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$
2	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$ ✓	$\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$ ✓
3	$\begin{pmatrix} 36 & 28 \\ 28 & 36 \end{pmatrix}$	$\begin{pmatrix} 112 & 104 \\ 104 & 112 \end{pmatrix}$
4	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$	$\begin{pmatrix} 656 & 640 \\ 640 & 656 \end{pmatrix}$
5	$\begin{pmatrix} 528 & 496 \\ 496 & 528 \end{pmatrix}$	$\begin{pmatrix} 3904 & 3872 \\ 3872 & 3904 \end{pmatrix}$

how? GDC? E again, say so!

It is more difficult to spot any patterns in these results. To help, the numbers obtained are broken down in the tables below

For P

n	1st element	2nd element
1	3	1
2	$10 = 3 \times 3 + 1$	$6 = 3 + 1 \times 3$
3	$36 = 10 \times 3 + 6$	$28 = 10 + 6 \times 3$
4	$136 = 36 \times 3 + 28$	$120 = 36 + 28 \times 3$
5	$528 = 136 \times 3 + 120$	$496 = 136 + 120 \times 3$

Let $P^n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix}$

then these patterns can be written as

$a_1 = a_1$ $b_1 = b_1$

$a_2 = 3a_1 + b_1$ $b_2 = a_1 + 3b_1$

$a_3 = 3a_2 + b_2$ $b_3 = a_2 + 3b_2$

$a_4 = 3a_3 + b_3$ $b_4 = a_3 + 3b_3$

which leads to the general formula

$$P^n = \begin{pmatrix} 3a_{n-1} + b_{n-1} & a_{n-1} + 3b_{n-1} \\ a_{n-1} + 3b_{n-1} & 3a_{n-1} + b_{n-1} \end{pmatrix}$$

For S

n	1st element	2nd element
1	4	2
2	$20 = 4 \times 4 + 2 \times 2$	$16 = 4 \times 2 + 2 \times 4$
3	$112 = 20 \times 4 + 16 \times 2$	$104 = 20 \times 2 + 16 \times 4$
4	$656 = 112 \times 4 + 104 \times 2$	$640 = 112 \times 2 + 104 \times 4$
5	$3904 = 656 \times 4 + 640 \times 2$	$3872 = 656 \times 2 + 640 \times 4$

This generalises to

$$S^n = \begin{pmatrix} 4a_{n-1} + 2b_{n-1} & 2a_{n-1} + 4b_{n-1} \\ 2a_{n-1} + 4b_{n-1} & 4a_{n-1} + 2b_{n-1} \end{pmatrix}$$

3. If $k = 4$, $\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} = R$

$$R^n = R^{n-1}R$$

$$= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 5a_{n-1} + 3b_{n-1} & 3a_{n-1} + 5b_{n-1} \\ 3a_{n-1} + 5b_{n-1} & 5a_{n-1} + 3b_{n-1} \end{pmatrix}$$

✓ good analysis! C

If $k = 5$ the matrix is $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} = Q$

$$Q^n = Q^{n-1} Q$$
$$= \begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix} \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 6a_{n-1} + 4b_{n-1} & 4a_{n-1} + 6b_{n-1} \\ 4a_{n-1} + 6b_{n-1} & 6a_{n-1} + 4b_{n-1} \end{pmatrix}$$

This suggests the general result

$$\begin{pmatrix} k+1 & k-1 \\ k-1 & k+1 \end{pmatrix}^n = \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_n \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_n \end{pmatrix}$$

4. Let $k = 2$, $n = -1$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3/8 & -\frac{1}{8} \\ -\frac{1}{8} & 3/8 \end{pmatrix}$$

how did you obtain $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$? B

Let $k = 2$, $n = -2$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-2} = \begin{pmatrix} 5/32 & -\frac{3}{32} \\ -3/32 & 5/32 \end{pmatrix}$$

$$3 \cdot \frac{5}{32} + -\frac{3}{32} = \frac{12}{32} = 3/8$$

$$\frac{5}{32} + 3\left(-\frac{3}{32}\right) = -\frac{4}{32} = -1/8$$

So the result also holds for negative values of n.

$n = 0$. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$3 \times 1 + 0 \quad = 3 \qquad 1 + 3 \times 0 \quad = 1$

So the result holds for $n = 1$

$3 \times \frac{3}{8} + -\frac{1}{8} = \frac{8}{8} = 1$

$\frac{3}{8} + 3\left(-\frac{1}{8}\right) = \quad 0 = 0$

So result also holds for $n = 0$.

$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^n$ is undefined if n is not an integer.

The result holds for $n \in \mathbb{Z}$

Suppose k is negative
Let $k = -1$ giving the matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

and let $k = -2$ giving the matrix $\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$

The table shows the first few results

n	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}^n$	$\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}^n$
1	$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$	$\begin{matrix} 10 & 6 \\ 6 & 10 \end{matrix}$
2	$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$\begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$
3	$\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$	$\begin{pmatrix} -28 & -36 \\ -36 & -28 \end{pmatrix}$
4	$\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$	$\begin{pmatrix} 136 & 120 \\ 120 & 136 \end{pmatrix}$
5	$\begin{pmatrix} -32 & -32 \\ -32 & 0 \end{pmatrix}$	$\begin{pmatrix} -496 & -528 \\ -528 & -496 \end{pmatrix}$

These results fit the general formula so it seems that the result holds for negative values of k.

Let $k = \frac{1}{2}$

$$\begin{array}{cc} n & \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}^n \\ 1 & \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix} \\ 2 & \begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix} \\ 3 & \begin{pmatrix} 9/2 & -7/2 \\ -7/2 & 9/2 \end{pmatrix} \\ 4 & \begin{pmatrix} 17/2 & -15/2 \\ -15/2 & 17/2 \end{pmatrix} \end{array}$$

good discussion of scope and limitations! (D)

Substituting into the general formula

$\frac{3}{2}\frac{3}{2} + -\frac{1}{2}\cdot-\frac{1}{2} = \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$

$-\frac{1}{2}\cdot\frac{3}{2} + \frac{3}{2}\cdot-\frac{1}{2} = -\frac{6}{4} = \frac{3}{2}$

suggests that the result will also hold for rational values of k.

Let $k = \sqrt{2}$

pursue this! (D) $K \in \mathbb{R}$ works!

$$\begin{array}{cc} n & \begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}^n \\ 1 & \begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix} \\ 2 & \begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}\begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix} = 2\sqrt{3}+3 \\ 3 & \end{array}$$

??

The result holds for $n \in \mathbb{Z}$ and k rational.

5. This is easily justified by matrix multiplication.

$$\begin{pmatrix} a_{n-1} & b_{n-1} \\ b_{n-1} & a_{n-1} \end{pmatrix}\begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}$$

$$= \begin{pmatrix} (k+1)a_{n-1} + (k-1)b_{n-1} & (k-1)a_{n-1} + (k+1)b_{n-1} \\ (k-1)a_{n-1} + (k+1)b_{n-1} & (k+1)a_{n-1} + (k-1)b_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} (k+1) & (k-1) \\ (k-1) & (k+1) \end{pmatrix}^n.$$

Satisfactory informal justification D

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student B		
Title of task: Matrix powers		**Type:** I II
Date set:	**Date submitted:**	

A Use of notation and terminology 2 / 2

Good use of appropriate notation throughout.

B Communication 3 / 3

Well presented. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ result appears without working or explanation.

Overall communication, however, is clear and coherent.

C Mathematical process 5 / 5

Good analysis. Validity tested.

D Results 5 / 5

While the recursive statement does not give an explicit result, it is an acceptable general statement. Discussion of scope and limitations is incomplete, but correct, satisfactory informal justification.

E Use of technology 2 / 3

No evidence of use is provided. Limited use only. You could have explored irrational values of *k* with decimal approximations.

F Quality of work 1 / 2

Good work. An explicit generalized statement and consideration of irrational values of *k* would have demonstrated greater insight.

Stopping distances—student C

SL Type II

Stopping Distances

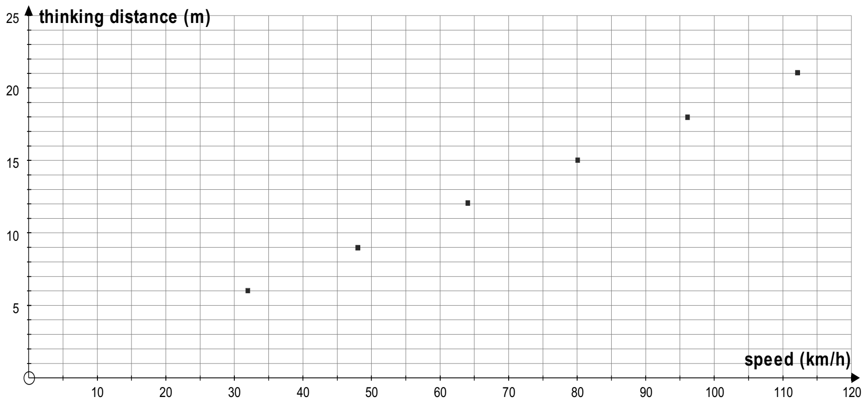
In this assignment I will be building up a model for the relationship between speed and overall stopping distance for cars.

The table below gives average times for Thinking Distance and Braking Distance at various speeds.

Speed (kmh^{-1})	Thinking distance (m)	Braking distance (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

Source:

Let s kmh^{-1} be the speed and d metres be the thinking distance. The graph below is of d against s.



This looks like a straight line. Considering the first two points, its gradient is

$\frac{9-6}{48-32}=\frac{3}{16}$. Also when the speed is zero the thinking distance will also be zero.

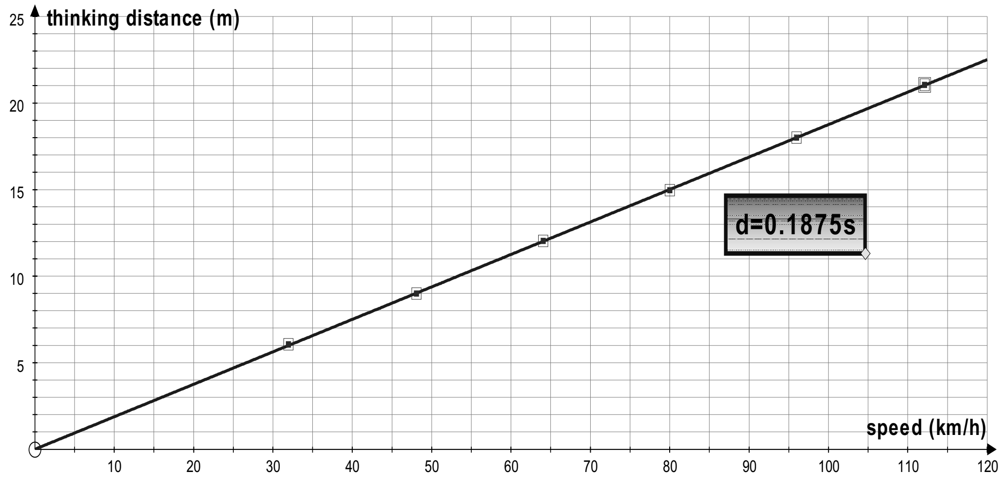
The equation of the line can now be found

$$d-0=\frac{3}{16}(s-0)$$

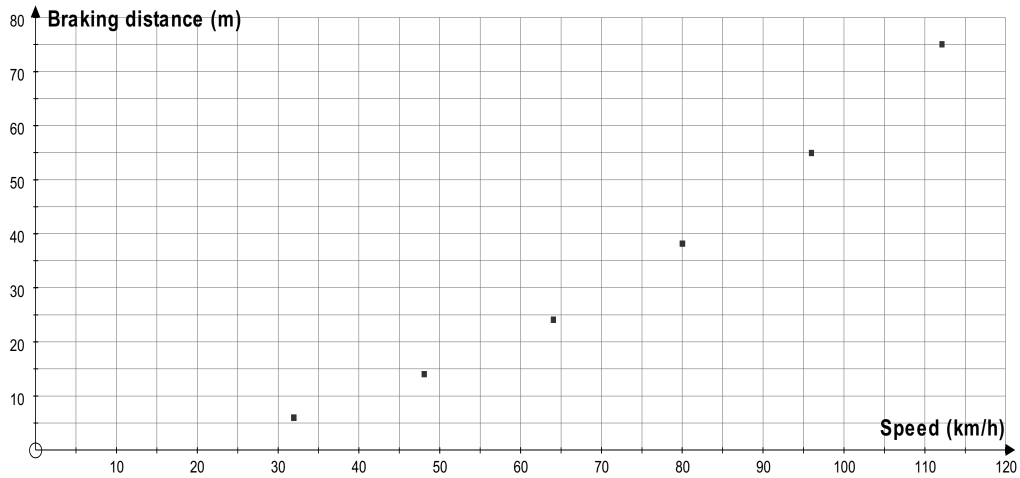
$$d=\frac{3}{16}s$$

1

The graph below shows the graph of this function and the original data. The fit is excellent.



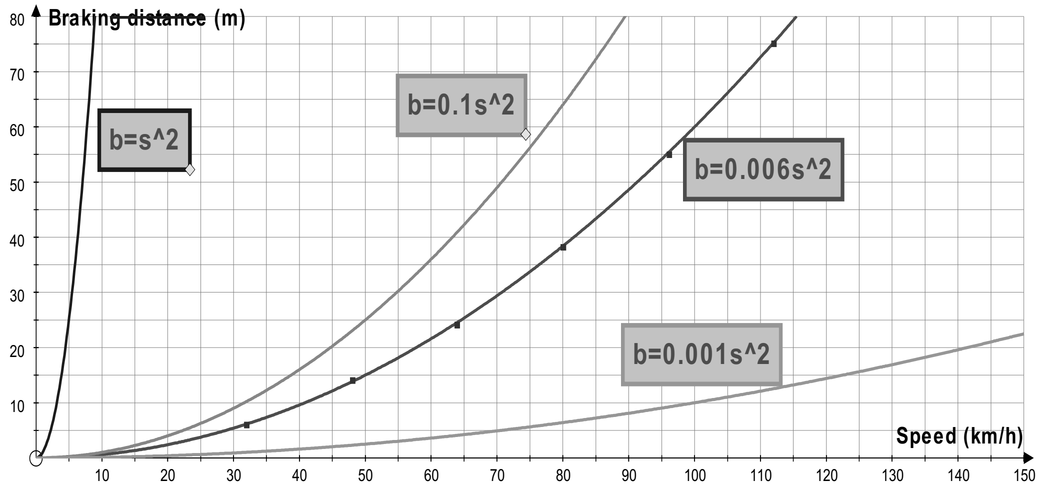
Now I will consider the braking distance data.
Let b metres be the braking distance. The graph below is of b against s.



This looks like the general shape of part of a quadratic function with its vertex at (0,0) The means that the function has the form $b = as^2$. Using Autograph, I tried to find a suitable value of *a*. As *a* changes the graph is stretched vertically with scale factor *a*. I first drew $b = s^2$ to see whether *a* was greater than or less than one. Clearly it is considerable less than one. Using trial and error I found that

2

0.006 was a suitable value of a. The diagram below shows a few of the values I tried.



So a suitable function to model braking distance is
$b = 0.006s$

Now I will try to find a model for the overall stopping distance.

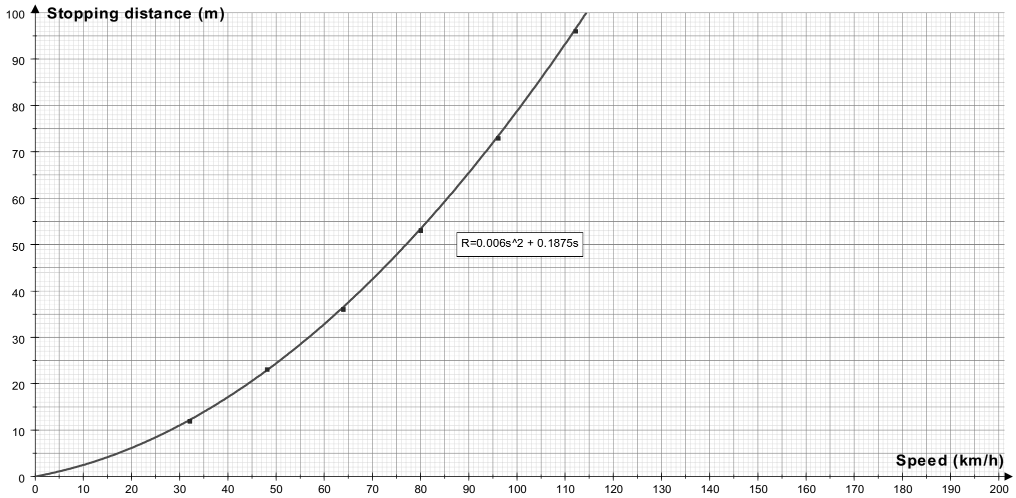
The table below gives the relevant values

Speed (kmh^{-1})	Stopping distance (m)
32	12
48	23
64	36
80	53
96	73
112	96

Since we add the thinking distance to the braking distance to get overall stopping distance we should add the expressions for these to get a formula for the stopping distance. The graph shows the data from the table above and the graph of the function.

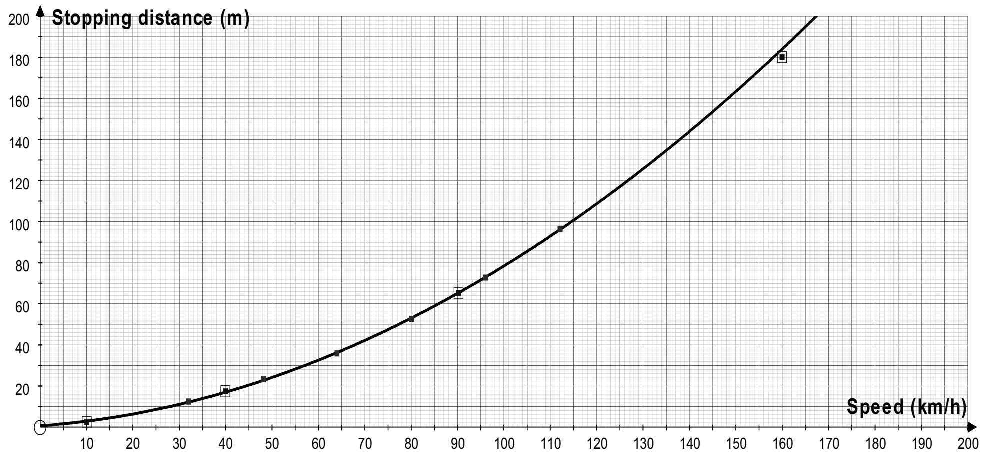
$$R = 0.006s^2 + \frac{3}{16}s\ .$$

3



The model fits the data well for the range of speeds given.

Additional data, including speeds outside this range is given.
The next graph includes this data.



It can be seen that the model still appears to be good. The value for a speed of 160kmh^{-1} is slightly off the curve and so the fit may not be so good for higher speeds. More data would have been helpful to test this further.

4

As Autograph can fit curves to data I thought it would be interesting to see what function it came up with. The function it produced was

$$R = 0.006138s^2 + 0.1643s + 0.6\,.$$

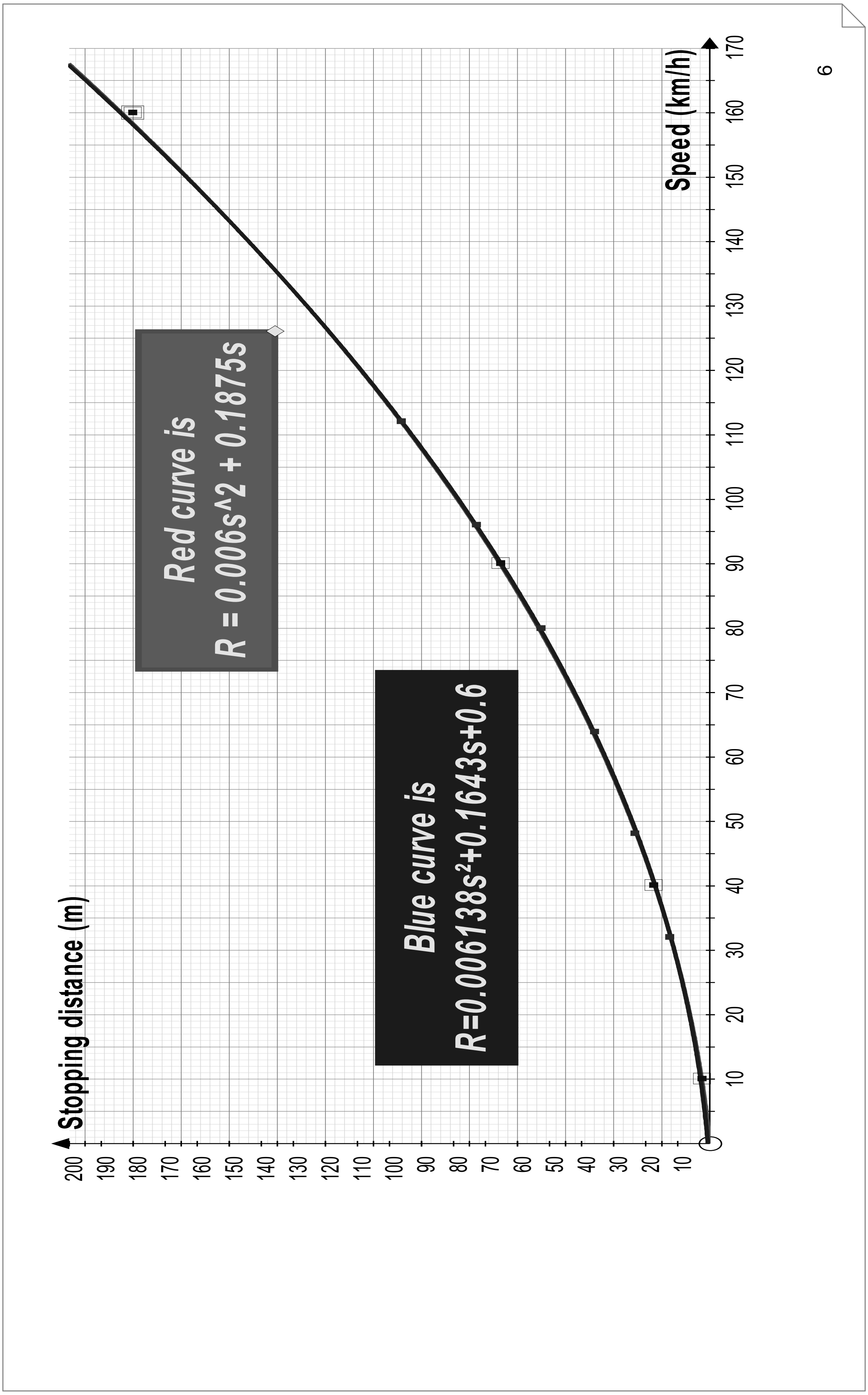
This is compared with the function I have found on the following graph. Even when enlarged the curves are almost identical, so both functions fit the data well. However, I prefer my model as it is built up of the two parts of the overall stopping distance and also it gives R=0 when s=0 which it what really happens.

The original data was for cars. It is likely that heavier vehicles would require longer stopping distances as their momentum would be greater for the same speed. So the model should only be used for cars. Even for cars other factors might affect the stopping distance. It is well known that when the roads are wet there is less friction between the tyres and the road surface and stopping takes longer distances. Rough surfaces would need less distance. Probably the data was collected from vehicles with brakes in top condition. Older vehicles with less good brakes would not perform so well. Who was driving the cars? Older people generally have slower reaction times so would need longer thinking distances.

The model fits the data well. So it will be useful for working out stopping distances for cars but only on the kind of road surface used when data was collected and with cars and drivers with ages similar to those that took part in the test.

Software used: Autograph (version3) www.autograph-maths.com

5


6
Stopping distance (m)
Speed (km/h)
Red curve is
R = 0.006s^2 + 0.1875s
Blue curve is
R=0.006138s²+0.1643s+0.6

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student C		
Title of task: Stoppping distances		**Type:** I (II)
Date set:	**Date submitted:**	

A Use of notation and terminology 2 / 2

Good overall. One small error where *s* is used instead of s^2 but this is allowed.

B Communication 3 / 3

The additional data (page 55) and the comparison of the two graphs (the model and the regression) should be placed in context. The general level of communication, however, is excellent.

C Mathematical process 5 / 5

Variables are defined. Parameter “a” is successfully determined. Model function fit is considered and model function is applied to additional data.

D Results 4 / 5

Reasonableness of the model had been carefully considered. Two models have been compared, but limitations have not been considered. The degree of accuracy used throughout is appropriate even though not discussed.

E Use of technology 3 / 3

Excellent use of Autograph® software. Graphs are used effectively to enhance understanding.

F Quality of work 2 / 2

Excellent work.

Modelling the amount of a drug in the bloodstream—student D

SL Type II

In this assignment I will investigate the absorption of a drug into the human bloodstream by creating a model function that matches the supplied data. I will also investigate how regular doses of the drug cause the overall quantity of drug in the bloodstream to change with time.

To creat a model function I will first create a table of values by reading coordinates from the supplied graph. I will use the variable t to represent the time elapsed, in hours. I will use the variable A to represent the amount of drug in the bloodstream, in µg. Clearly t and A will take on only positive or 0 values.

Part A

The following data was read from the supplied graph, and is accurate to the nearest tenth of a unit. Once again, t is in hours, and A is in µg.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10
A	10	9	8.3	7.8	7.2	6.7	6	5.3	5	4.5	4.3	4	3.7	3	2.8	2.5	2.5	2.1	1.9	1.8	1.5

The task states that the rate of decrease is approximately proportional to the amount of drug remaining in the bloodstream. This suggest that each subsequent amount is a fraction of the previous measured amount. This would mean that this is a decay model. Such situations can be modelled by a function of the form $A = A_0 e^{\lambda t}$, where A_0 is the initial quantity of drug administered. I know that A_0 must be 10µg, since this is the value of A at $t = 0$. Thus we have $A = 10e^{\lambda t}$. We must now determine a suitable constant, λ.

If I substitute the coordinates (t, A) from the graph I can obtain individual values of λ for each pair as follows;

$(t, A) = (3.0, 6)$

$$6 = 10e^{3\lambda}$$

$$0.6 = e^{3\lambda}$$

$$\ln(0.6) = 3\lambda$$

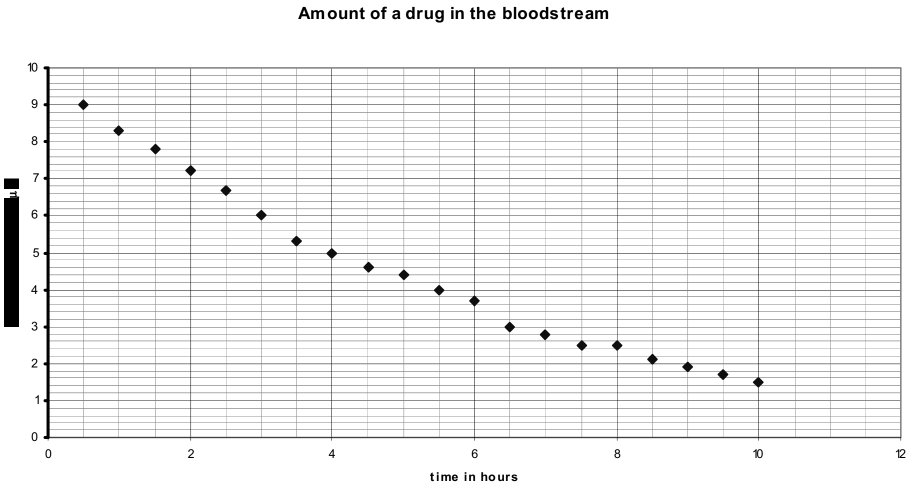
$$\lambda = \frac{\ln(0.6)}{3} \cong -0.170 \quad \text{3 s.f.}$$

However, this is only a single instance, and will not be representative of the actual value of λ. A better estimate can be obtained by calculating the value of λ for each pair of coordinates, and then averaging these values. By using the same method as above I calculated λ-values for each point. The calculated values of λ are presented below, rounded to 3 s.f.. Note that a value for λ cannot be found for $t = 0$ since division by 0 is undefined. The average value of λ was found using more accurate calculated values and then rounding at the end.

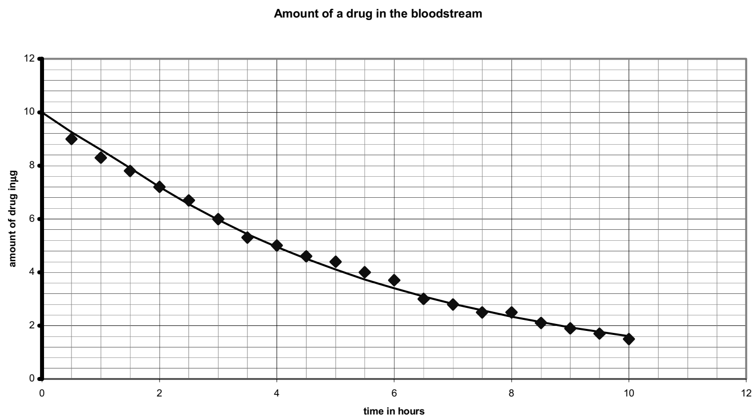
λ = {NA, -0.211, -0.186, -0.166, -0.164, -0.160, -0.170, -0.181, -0.173, -0.177, -0.169, -0.167, -0.166, -0.185, -0.182, -0.185, -0.173, -0.184, -0.185, -0.180, -0.190}

$\bar{\lambda} \cong -0.178$

I can now use the function $A = 10e^{-0.178t}$ to approximately model the data. The graph below shows the set of points (t, A) as supplied. The second graph shows the data and the model function.



Data Plot of Amount *vs* Time

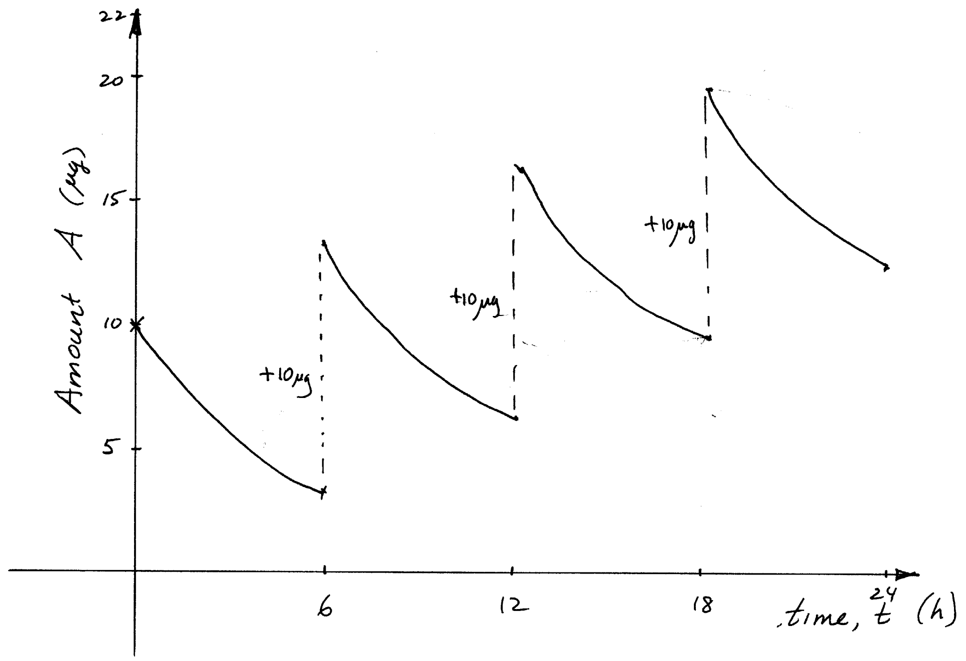


Data Plot and $A = 10e^{-0.178t}$

The function fits the data quite well and some discrepancy is to be expected due to the approximate measures involved. The function follows the same behaviour as the data, although the real-life situation does not include negative values of time. The function continues past $t = 10$ and approximates what will happen to the amount of drug remaining in the bloodstream as time continues to pass. Note that the amount is tending towards 0, although the model function does not allow the amount to ever really reach 0. In reality I would expect that the drug eventually disappears from the bloodstream altogether.

Part B

Now I will investigate how the amount of drug in the bloodstream changes when a 10μg dose is taken every 6 hours. Clearly the amount of drug in the blood will decrease according to the model above, but jump up 10 units every 6 hours. I expect that the graph would look like the sketch below.



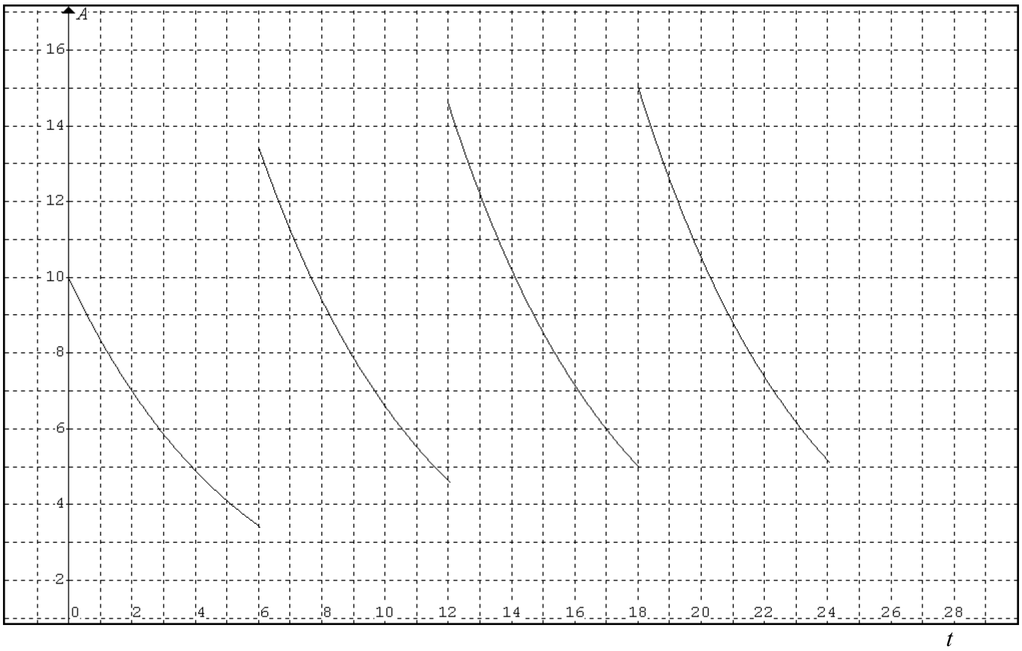
Here I assume that the new dose is immediately and completely absorbed into the bloodstream, or else the increase in the amount would be gradual. I would also expect that the amount remaining after 6 hours from the first dose would add to the total amount in the blood. So, in fact, the amount after 6 hours (and for subsequent additional doses) would include the residual amount from the first dose plus the amount left from the second dose. Thus the pieces of the graph above would be comprised of a sum of versions of the model function, adjusted for a time translation.

That is, for $6 \le t < 12$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)}$

and for $12 \le t < 18$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)} + 10e^{-.178(t-12)}$

and for $18 \le t < 24$ $\quad A = 10e^{-.178t} + 10e^{-.178(t-6)} + 10e^{-.178(t-12)} + 10e^{-.178(t-18)}$

Using Graphmatica, I graphed these functions. The results are shown below.

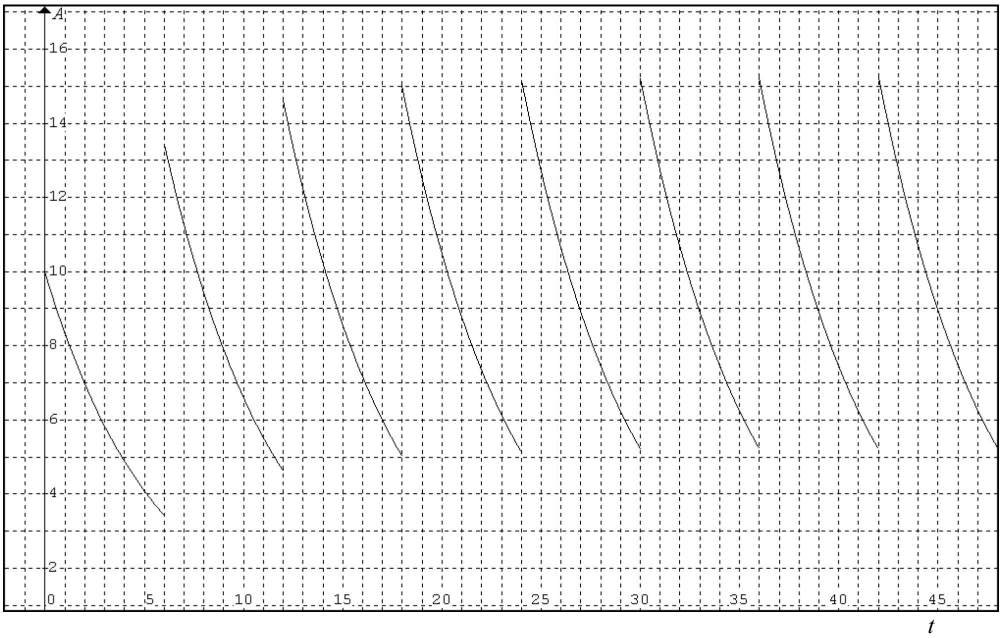


y=10*e^(-.178(x-18))+10*e^(-.178(x-12))+10*e^(-.178(x-6))+10*e^(-.178x) {18,24}

The minimum value is approximately 3.8μg, while the maximum is approximately 15μg.

If no further doses were taken I would expect that the amount of drug in the bloodstream would decrease in a fashion similar to how the original model function decreased, as there is no new supply. Eventually there would be no drug left in the blood system.

If doses were continued to be taken every 6 hours, it appears that the maximum amount would stabilize, as would the minimum. The cummulative effect of the previous doses would decrease as time goes by due to the exponential rate of decay. This would mean that the minimum (and thus the maximum since this is just the previous minimum increased by 10) would tend to a stable value. I have graphed the longer term behaviour below, and this confirms my conjecture.



This behaviour makes sense medically, as a doctor would want the level of a medication in a patient to stabilize within certain safe and effective values, and not fluctuate wildly over time.

Mathematics SL: The portfolio **Feedback to student**		**Form B**
Name: Student D		
Title of task: Modelling the amount of a drug in the bloodstream		**Type:** I II
Date set:	**Date submitted:**	

A Use of notation and terminology 2 / 2

Correct throughout including appropriate use of “approximately equals” sign

B Communication 3 / 3

Excellent communication. Explanations are clear and well-supported by good graphs. The piece can be read without reference to the statement of the task.

C Mathematical process 5 / 5

Variables are clearly defined. The parameters of the function are linked to the context. A suitable model is formulated and student considers how well it fits the data. The initial model is adapted for repeated doses.

D Results 4 / 5

The results are correctly interpreted in context and the degree of accuracy used is appropriate. Some limitations of the model are discussed on page 61. Some major simplifications are mentioned (immediate absorption) but not referred to when considering the overall quality of the model. The final part is imprecise and not completely correct.

E Use of technology 3 / 3

Developing the piecewise function is critical to the development of the task.

F Quality of work 1 / 2

Not quite enough for a level 2. A more precise mathematical analysis of final graph would have made this a level 2, for example, an attempt to fit a function to a graph of the maximum values.

