MARKSCHEME

May 2007

MATHEMATICS

Higher Level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $f'(x) = 2\cos(5x-3)\ \ 5 \quad = 10\cos(5x-3)$ ***A1***

  Award ***A1*** for $2\cos(5x-3)\ \ 5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)****. Award the marks as usual then write* ***(AP)*** *against the answer. On the* **front** *cover write –1****(AP).*** *Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

SECTION A

Statistics and probability

1. (a) Mean $= \dfrac{1\times 19 + 2\times 34 + \ldots + 5\times 4}{100}$ ***(M1)***

$= 2.16$ ***A1*** ***N2***

[2 marks]

(b) (i) H_0 : Poisson law provides a suitable model ***A1***

H_1 : Poisson law does not provide a suitable model ***A1***

(ii) The expected frequencies are

Number of eggs	Observed frequency	Expected frequency	
0	10	11.533	***A1***
1	19	24.910	***A1***
2	34	26.903	***A1***
3	23	19.370	***A1***
4	10	10.460	***A1***
5 or more	4	6.824	***A1***

Note: Accept expected frequencies rounded to a minimum of three significant figures.

$\chi^2 = \dfrac{(10-11.533)^2}{11.533} + \ldots + \dfrac{(4-6.824)^2}{6.824}$ ***(M1)(A2)***

$= 5.35$ (accept 5.33 and 5.34) ***A2***

$\nu = 4$ (6 cells – 2 restrictions) ***A1***

Note: If candidates have combined rows allow ***FT*** on their value of ν .

Critical value $\chi^2 = 13.277$ ***A1***

Because $5.35 < 13.277$,the Poisson law does provide a suitable model. ***R1*** ***N0***

[16 marks]

Total [18 marks]

2. H_0 : Mean difference $= 0$; H_1 : Mean difference $\neq 0$

Finding the differences. ***M1***

1.5 0.3 2.9 1.7 –1.4 2.5 1.1 1.7 0.6 –0.4 ***A2***

EITHER

$\sum d = 10.5$ **OR** $\bar{d} = 1.05$ ***(A1)***

$\sum d^2 = 26.47$ ***(A1)***

$s_{n-1}^2 = \frac{26.47}{9} - \frac{10.5^2}{90} = 1.71611\ldots$ ***(M1)(A1)***

$$t = \frac{\frac{10.5}{10}}{\sqrt{\frac{1.71611\ldots}{10}}} = 2.53(464\ldots)$$ ***(M1)A1***

Degrees of freedom $= 9$ ***A1***

Critical value $= 2.262$ ***A1***

$2.53 > 2.262$ so the claim is not justified. (allow ***FT*** on their values) ***R1*** ***N0***

OR

p-value $= 0.0320$ (accept 0.03) ***A8***

$0.0320 < 0.05$ so the claim is not justified. (allow ***FT*** on their values) ***R1*** ***N0***

[12 marks]

3. $\mathrm{P}\left(Z > \frac{10.2 - \mu}{\sigma}\right) = 0.05$ ***(M1)***

$10.2 = \mu + 1.6449\sigma$ ***A1***

Standard Error of $\bar{X} = \frac{\sigma}{\sqrt{7}}$ ***(A1)***

$$\mathrm{P}\left(Z < \frac{6.1 - \mu}{\frac{\sigma}{\sqrt{7}}}\right) = 0.025$$ ***M1***

$6.1 = \mu - 1.96 \times \frac{\sigma}{\sqrt{7}}$ ***A1***

$10.2 - 6.1 = \sigma\left(1.6449 + \frac{1.96}{\sqrt{7}}\right)$ ***(M1)***

$\sigma = 1.72$ ***A1*** ***N0***

$\mu = 10.2 - 1.6449 \times 1.718\ldots$ ***(M1)***

$= 7.37$ ***A1*** ***N0***

[9 marks]

4. (a) Critical (region) ***A1***

[1 mark]

(b) (i) $\mathrm{P}(\text{Type I error}) = \mathrm{P}(X \geq 2 \mid b = 5)$ ***(M1)(A1)***

$$= \frac{5\times4\times3}{15\times14\times13} + 3\times\frac{5\times4\times10}{15\times14\times13} \quad \left(\text{or } \frac{\binom{5}{3}\binom{10}{0}}{\binom{15}{3}} + \frac{\binom{5}{2}\binom{10}{1}}{\binom{15}{3}} \right)$$ ***(A1)(A1)***

$$= 0.242 \quad \left(\frac{22}{91}\right)$$ ***A1***

(ii) $\mathrm{P}(\text{Type II error}) = \mathrm{P}(X < 2 \mid b = 9)$ ***(M1)(A1)***

$$= \frac{6\times5\times4}{15\times14\times13} + 3\times\frac{6\times5\times9}{15\times14\times13} \quad \left(\text{or } \frac{\binom{6}{3}\binom{9}{0}}{\binom{15}{3}} + \frac{\binom{6}{2}\binom{9}{1}}{\binom{15}{3}} \right)$$ ***(A1)(A1)***

$$= 0.341 \quad \left(\frac{31}{91}\right)$$ ***A1***

[10 marks]

Total [11 marks]

5. (a) X_1 is the number of trials up to the 1st success
X_2 is the number of subsequent trials up to the 2nd success
X_{20} is the number of subsequent trials up to the 20th success **R1**

$\sum_{i=1}^{20} X_i$ is the total number of trials up to the 20th success and therefore has a negative binomial distribution. **R1**

[2 marks]

(b) Mean $= \frac{20}{0.6} = 33.3 \quad \left(\frac{100}{3}\right)$ ***(M1)A1***

Variance $= \frac{20 \times 0.4}{0.6^2} = 22.2 \quad \left(\frac{200}{9}\right)$ ***(M1)A1***

[4 marks]

(c) $\mathrm{P}(Y = 30) = \binom{29}{19} \times 0.6^{20} \times 0.4^{10}$ ***M1(A2)***

$= 0.0768$ ***A1*** ***N2***

[4 marks]

Total [10 marks]

SECTION B

Sets, relations and groups

1. (a) Reflexive: aRa because $a^2 \equiv a^2$ (modulo 3) ***A1***

Symmetric: $aRb \Rightarrow a^2 - b^2$ divisible by 3

$\Rightarrow b^2 - a^2$ divisible by $3 \Rightarrow bRa$ ***M1A1***

Transitive: $aRb \Rightarrow a^2 - b^2 = 3m$, $(m \in \mathbb{Z})$ ***M1***

$bRc \Rightarrow b^2 - c^2 = 3n$, $(n \in \mathbb{Z})$ ***A1***

$\Rightarrow a^2 - c^2 = 3(m+n) \Rightarrow aRc$ ***A1***

[6 marks]

(b) The equivalence classes are

$3n-2$; $3n-1$; $n \in \mathbb{Z}^+$ ***A2***

$3n$; $n \in \mathbb{Z}^+$ ***A2***

[4 marks]

Total [10 marks]

2. (a) $a*b = a+b-1$; $b*a = b+a-1$ ***A1***

These are equal, therefore the operation is commutative. ***R1***

[2 marks]

(b) Closure: $a, b \in \mathbb{R} \Rightarrow a*b = a+b-1 \in \mathbb{R}$ ***R1***

Identity: For an identity *e* we require

$a*e = a+e-1 = a$ so $e = 1 \in \mathbb{R}$ ***M1A1***

Inverse: For an inverse of *a* we require

$a*a^{-1} = a+a^{-1}-1 = 1$ ***M1A1***

so $a^{-1} = 2-a \, (\in \mathbb{R})$ ***A1***

Associativity:

$a*(b*c) = a*(b+c-1)$ ***M1***

$= a+b+c-2$ ***A1***

$(a*b)*c = (a+b-1)*c$ ***M1***

$= a+b+c-2$ ***A1***

These are equal, therefore the operation is associative. ***A1***

The four conditions are satisfied so $\{\mathbb{R}, *\}$ is a group. ***R1***

[12 marks]

Total [14 marks]

3. (a) $p_1^{\,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$ ***A1***

$p_1^{\,3} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$ ***A1***

This shows that the first 3 elements cycle with order 3 and the last 2 with order 2. The order of p_1 is therefore 6. ***R2***

[4 marks]

(b) (i) $p_2 p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 & 5 \end{pmatrix}$ ***A2***

(ii) Attempting to find $p_1 p_2$ (as many terms as needed, *e.g.* $1 \Rightarrow 1$) ***M1***

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}$

They do not commute. ***R1***

[4 marks]

(c) $p_1^{\,2} p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$ ***A2***

$(p_1^{\,2} p_2)^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$ ***A2***

[4 marks]

Total [12 marks]

4. (a) (i) c is the identity of $\circ$ ***A1***
b is the identity of $\times$ ***A1***

(ii) For $\circ$, $a^2 = b^2 = d^2 = c$ ***(M1)***
So a, b, d have order 2, c order 1 ***A1***
For $\times$, $c^2 = b$, $a^4 = d^4 = b$ ***(M1)***
So a, d have order 4, c has order 2, b has order 1 ***A1***

[6 marks]

(b) (i) $\{a, c\}$; $\{b, c\}$; $\{c, d\}$ ***A1A1A1***

(ii) $\{b, c\}$ ***A1***

[4 marks]

(c) Post-multiply by the inverse of d under $\times$, *i.e.* a ***(M1)***
$[a \circ (x \times x)] \times d \times a = c \times a$ ***(A1)***
$a \circ (x \times x) = d$ ***A1***
Pre-multiply by the inverse of a under $\circ$, *i.e.* a ***(M1)***
$a \circ a \circ (x \times x) = a \circ d$ ***(A1)***
$x \times x = b$ ***A1***
We now see by inspection that $x = b$ or $x = c$ ***A1A1***

[8 marks]

Total [18 marks]

5. Let $x \in A$ ***M1***
Then $x \in A \cup B = A \cap B$ ***A1***
So $x \in B$ and $A \subseteq B$ ***A1***
Let $x \in B$ ***M1***
Then $x \in A \cup B = A \cap B$
So $x \in A$ and $B \subseteq A$ ***A1***
Since $A \subseteq B$ and $B \subseteq A$ (or stated in words), it follows that $A = B$. ***R1***

[6 marks]

SECTION C

Series and differential equations

1. (a) (i) $\lim_{x\to1}\frac{\ln x^2}{x-1}=\lim_{x\to1}\frac{\frac{2}{x}}{1}$ ***M1A1***

$=2$ ***A1*** ***N0***

(ii) $\lim_{x\to0}\frac{\tan^2 x}{1-\cos x}=\lim_{x\to0}\frac{2\tan x\sec^2 x}{\sin x}$ ***M1A1***

EITHER

$=\lim_{x\to0}\frac{2\sec^4 x+4\tan^2 x\sec^2 x}{\cos x}$ ***M1A1***

$=2$ ***A1*** ***N0***

OR

$=\lim_{x\to0}2\sec^3 x$ ***M1A1***

$=2$ ***A1***

[8 marks]

(b) The argument is incorrect because the denominator is not zero when $x=3$. ***R2***

[2 marks]

Total [10 marks]

2. **METHOD 1**

$f(x) = \mathrm{e}^{\sin x}$, $f(0) = 1$ *A1*

$f'(x) = (\cos x)\mathrm{e}^{\sin x}$

$f'(0) = 1$ *A1*

$f''(x) = -(\sin x)\mathrm{e}^{\sin x} + (\cos^2 x)\mathrm{e}^{\sin x}$ *M1*

$f''(0) = 1$ *A1*

$f'''(x) = -(\cos x)\mathrm{e}^{\sin x} - (\sin x \cos x)\mathrm{e}^{\sin x} + (\cos^3 x)\mathrm{e}^{\sin x} - 2(\sin x \cos x)\mathrm{e}^{\sin x}$ *M1*

$f'''(0) = 0$ *A1*

$a = 1,\ b = 1,\ c = \frac{1}{2},\ d = 0$ *A2* *N0*

[8 marks]

METHOD 2

$\mathrm{e}^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2} + \frac{\sin^3 x}{6} + \ldots$ *(M1)(A1)*

$= 1 + (x - \frac{x^3}{6} + \ldots) + \frac{1}{2}\left(x - \frac{x^3}{6} + \ldots\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{6} + \ldots\right)^3 + \ldots$ *M1A1*

$= 1 + x - \frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ *M1A1*

$= 1 + x + \frac{x^2}{2} + 0x^3 + \ldots$ *A2* *N0*

$(a = 1,\ b = 1,\ c = \frac{1}{2},\ d = 0)$

[8 marks]

3. (a) **EITHER**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} < \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which is convergent}$$ ***M1A1***

The given series is therefore convergent using the comparison test. ***R1***

[3 marks]

OR

$$\lim_{n\to\infty} \frac{1}{n(n+2)} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n}{n+2} = 1$$ ***M1A1***

The given series is therefore convergent using the limit comparison test. ***R1***

[3 marks]

(b) (i) Let $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$ $\left(= \frac{A(n+2)+Bn}{n(n+2)} \right)$ ***(M1)***

Substituting values for n

$A = \frac{1}{2}$ ***(A1)***

$B = -\frac{1}{2}$ ***(A1)***

$\frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$ ***A1***

(ii) Using partial fractions

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{\frac{1}{2}}{1} - \frac{\frac{1}{2}}{3} + \frac{\frac{1}{2}}{2} - \frac{\frac{1}{2}}{4} + \frac{\frac{1}{2}}{3} - \frac{\frac{1}{2}}{5} + \frac{\frac{1}{2}}{4} - \frac{\frac{1}{2}}{6} + \frac{\frac{1}{2}}{5} - \frac{\frac{1}{2}}{7} + \ldots$$ ***M1A1***

Recognizing the cancellation (in the telescoping series) (*e.g.* crossing out).***R1***

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$ ***A2***

[9 marks]

Total [12 marks]

4. (a) $I = \int \sin x \cos x \, \mathrm{e}^{-\sin x} \, \mathrm{d}x$

For a reasonable attempt at integration by parts. ***(M1)***

$u = \sin x \qquad v = -\mathrm{e}^{-\sin x}$

$\mathrm{d}u = \cos x \, \mathrm{d}x \qquad \mathrm{d}v = \cos x \, \mathrm{e}^{-\sin x} \mathrm{d}x$ ***(A1)***

$I = -\sin x \, \mathrm{e}^{-\sin x} + \int \cos x \, \mathrm{e}^{-\sin x} \, \mathrm{d}x$ ***A1A1***

$= -\sin x \, \mathrm{e}^{-\sin x} - \mathrm{e}^{-\sin x} + C$ ***AG***

[4 marks]

(b) $\mathrm{IF} = \mathrm{e}^{\int -\cos x \, \mathrm{d}x}$ ***(M1)(A1)***

$= \mathrm{e}^{-\sin x}$ ***A1***

[3 marks]

(c) $\mathrm{e}^{-\sin x} \dfrac{\mathrm{d}y}{\mathrm{d}x} - y \cos x \, \mathrm{e}^{-\sin x} = \sin x \cos x \, \mathrm{e}^{-\sin x}$ ***M1A1***

$\mathrm{e}^{-\sin x} y = \int \sin x \cos x \, \mathrm{e}^{-\sin x} \, \mathrm{d}x$ ***M1A1***

$\mathrm{e}^{-\sin x} y = -\sin x \, \mathrm{e}^{-\sin x} - \mathrm{e}^{-\sin x} + C$ ***A1***

Substituting $x = 0$ and $y = -2 \Rightarrow$ ***(M1)***

$-2 = 0 - 1 + C$

$-1 = C$ ***(A1)***

so $\mathrm{e}^{-\sin x} y = -\sin x \, \mathrm{e}^{-\sin x} - \mathrm{e}^{-\sin x} - 1$ ***A1***

$y = -\sin x - 1 - \mathrm{e}^{\sin x}$ ***A1***

[9 marks]

Total [16 marks]

5. Using the ratio test ***(M1)***

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\sin\left(\frac{\pi}{n+1}\right)x^{n+1}}{\sin\left(\frac{\pi}{n}\right)x^n}\right|$$ ***A1***

$$= |x|\lim_{n\to\infty}\left|\frac{\sin\left(\frac{\pi}{n+1}\right)}{\sin\left(\frac{\pi}{n}\right)}\right|$$ ***A1***

$$= |x|$$ ***A1***

The series is convergent for $|x| < 1$. ***A1***

When $x = 1$, the series is $\sum_{n=1}^{\infty}\sin\left(\frac{\pi}{n}\right)$ ***M1***

Using the limit comparison test with the divergent series $\sum_{n=1}^{\infty}\frac{1}{n}$ ***M1***

$$\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} = \pi$$ ***A1***

The series is divergent. ***R1***

When $x = -1$, the series is $\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{\pi}{n}\right)$ ***M1***

The series is alternating (after the first term). ***R1***
The general term is decreasing in magnitude and tending to zero as $n \to \infty$. ***R1***
The series is convergent. ***R1***

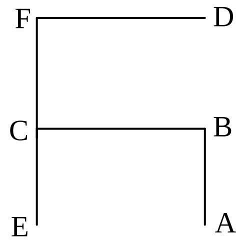
The interval of convergence of the original series is $-1 \le x < 1$. ***A1***

[14 marks]

SECTION D

Discrete mathematics

1. (a) The edges are introduced in the following order:
FD, FC, CB, BA, CE *A2A2A2A2A2*



A2

[12 marks]

(b) UB $=$ Twice weight of minimal spanning tree $= 2 \times 38 = 76$ *A2*

[2 marks]

Total [14 marks]

2. (a) Using Euclid's Algorithm,

$73 = 43 + 30$ *A1*
$43 = 30 + 13$ *A1*
$30 = 2 \times 13 + 4$ *A1*
$13 = 3 \times 4 + 1$ *A1*
The gcd is therefore 1 *R1*

[5 marks]

(b) (i) Working backwards, *(M1)*

$1 = 13 - 3 \times 4$
$= 13 - 3 \times (30 - 2 \times 13)$ *(A1)*
$= 7 \times 13 - 3 \times 30$
$= 7 \times (43 - 30) - 3 \times 30$ *(A1)*
$= 7 \times 43 - 10 \times 30$
$= 7 \times 43 - 10 \times (73 - 43)$ *(A1)*
$= -10 \times 73 + 17 \times 43$ *A1*

Therefore,
$7 \times 17 \times 43 - 7 \times 10 \times 73 = 7$ *M1A1*

The general solution is therefore
$x = 119 - 73k$, $y = -70 + 43k$, (where $k \in \mathbb{Z}$) *M1A1*

(ii) A graphical approach or trying successive values, gives $k = 2$ *M1*
and $(x, y) = (-27, 16)$ *A1* *N1*

[11 marks]

Total [16 marks]

3. (a) (i) The two vertices of odd degree are E and D. ***A1A1***

(ii) The shortest path between them is EBD. ***(M1)A1***

(iii) Each of the edges EB, BD will be traversed twice. ***(R2)***
One such path is ABEFGCBDBEGDFCA. ***A3***

(iv) Total weight = sum of all weights + two repeated edges ***M1***
$= 32 + 4 = 36$ ***A1*** ***N2***

[11 marks]

(b) One such cycle is ABEFDGCA. ***A2***

[2 marks]

Total [13 marks]

4. By Fermat's little theorem
$x^6 \equiv 1 \pmod 7$ so $x^{12} \equiv 1 \pmod 7$ ***A1A1***
$x^{12} + 1 \equiv 2 \pmod 7$ and cannot therefore equal $7y$. ***A1A1***
The above only applies when x is not a multiple of 7. ***R1***
Suppose now that x is a multiple of 7. Then, ***M1***
$x^{12} \equiv 0 \pmod 7$ ***A1***
so $x^{12} + 1 \equiv 1 \pmod 7$ and cannot therefore equal $7y$. ***A1***
Thus, in all cases, the equation has no solution. ***R1***

[9 marks]

5. (a) K' is a graph (with the same number of vertices as K) which has an edge between any two vertices if and only if K does not. ***A1***

[1 mark]

(b) The complete graph on 6 vertices is regular of degree 5. ***(A1)***
The degree of a vertex in K + the degree of a vertex in $K' = 5$ ***A1***
So if the degree of the vertex in K is even, it is odd in K', and vice versa. ***R1***
If K has m vertices of odd degree then K' has $6 - m$ vertices of odd degree. ***R1***
A graph has an Eulerian trail if and only if the number of vertices of odd degree is ≤ 2. ***(A1)***
At least one of m and $6 - m$ is greater than 2. ***R1***
Therefore they cannot both contain an Eulerian trail. ***R1***

[7 marks]

Total [8 marks]