IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

N05/5/MATME/SP2/ENG/TZ0/XX

88057302

**MATHEMATICAL METHODS**
**STANDARD LEVEL**
**PAPER 2**

Friday 4 November 2005 (morning)

2 hours

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INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

*Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.*

## SECTION A

*Answer all* ***five*** *questions from this section.*

**1.** *[Maximum mark: 12]*

(i) Two weeks after its birth, an animal weighed 13 kg. At 10 weeks this animal weighed 53 kg. The increase in weight each week is constant.

(a) Show that the relation between $y$, the weight in kg, and $x$, the time in weeks, can be written as $y = 5x + 3$. *[2 marks]*

(b) Write down the weight of the animal at birth. *[1 mark]*

(c) Write down the weekly increase in weight of the animal. *[1 mark]*

(d) Calculate how many weeks it will take for the animal to reach 98 kg. *[2 marks]*

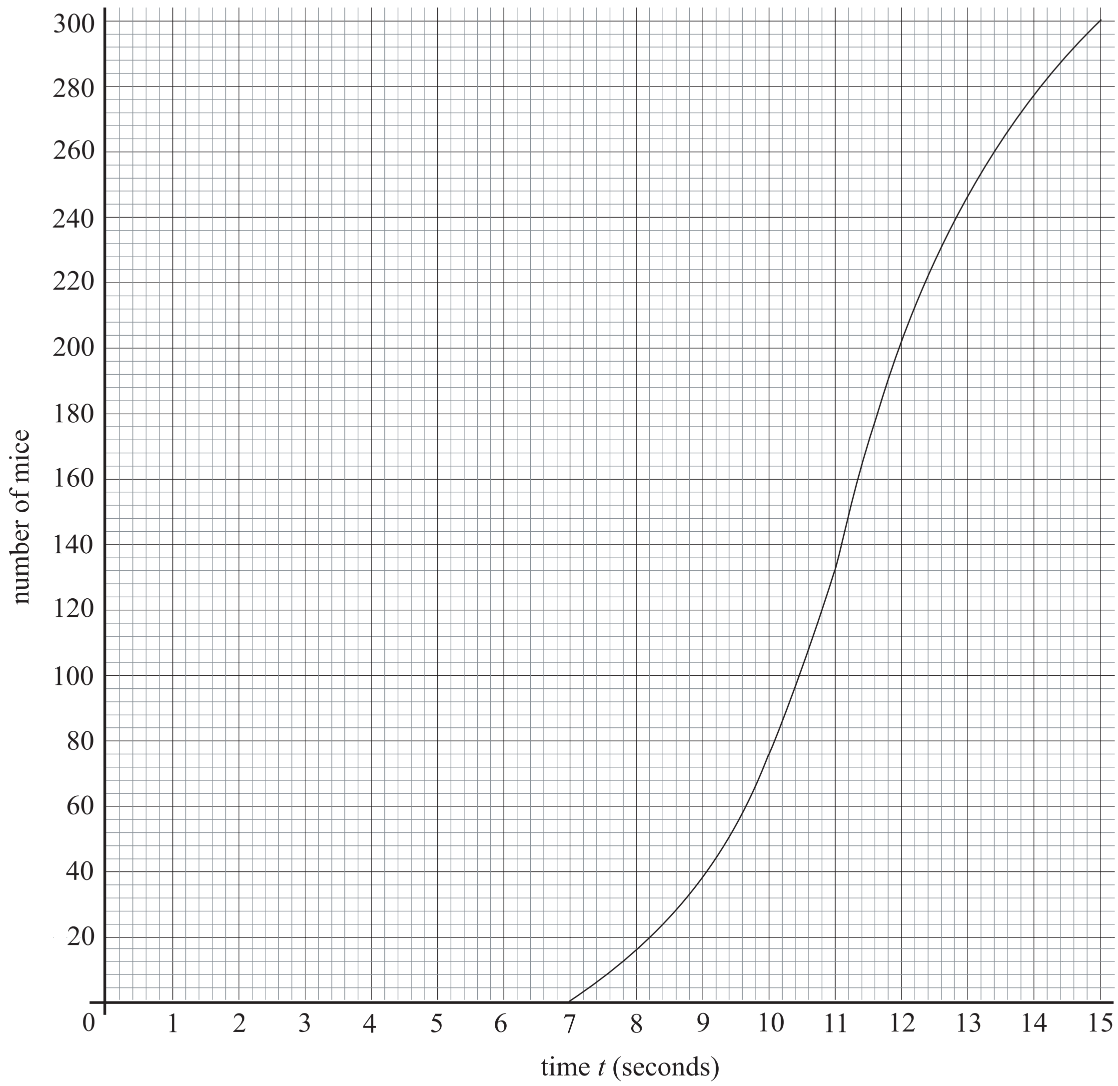
(ii) In the research department of a university, 300 mice were timed as they each ran through a maze. The results are shown in the cumulative frequency diagram opposite.

(a) How many mice complete the maze in less than 10 seconds? *[1 mark]*

(b) Estimate the median time. *[1 mark]*

*(This question continues on the following page)*

*(Question 1 (ii) continued)*




(c) Another way of showing the results is the frequency table below.

| Time $t$ (seconds) | Number of mice |
|---|---|
| $t < 7$ | 0 |
| $7 \le t < 8$ | 16 |
| $8 \le t < 9$ | 22 |
| $9 \le t < 10$ | $p$ |
| $10 \le t < 11$ | $q$ |
| $11 \le t < 12$ | 70 |
| $12 \le t < 13$ | 44 |
| $13 \le t < 14$ | 31 |
| $14 \le t < 15$ | 23 |

(i) Find the value of $p$ and the value of $q$.

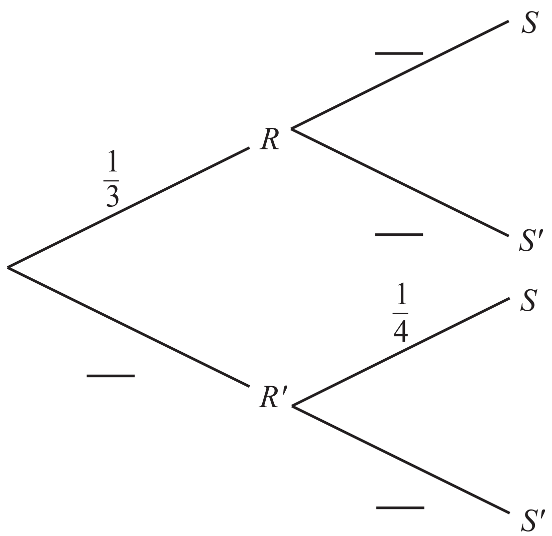
(ii) Calculate an estimate of the mean time. *[4 marks]*

**2.** *[Maximum mark: 10]*

The following probabilities were found for two events $R$ and $S$.

$$\mathrm{P}(R)=\frac{1}{3},\ \mathrm{P}(S\mid R)=\frac{4}{5},\ \mathrm{P}(S\mid R')=\frac{1}{4}.$$

(a) **Copy** and **complete** the tree diagram.




*[3 marks]*

(b) Find the following probabilities.

(i) $\mathrm{P}(R\cap S)$.

(ii) $\mathrm{P}(S)$.

(iii) $\mathrm{P}(R\mid S)$. *[7 marks]*

**3.** *[Maximum mark: 10]*

Consider the function $f(x) = \frac{16}{x-10} + 8,\ x \neq 10$.

(a) Write down the **equation** of

(i) the vertical asymptote;

(ii) the horizontal asymptote. *[2 marks]*

(b) Find the

(i) $y$-intercept;

(ii) $x$-intercept. *[2 marks]*

(c) Sketch the graph of $f$, clearly showing the above information. *[4 marks]*

(d) Let $g(x) = \frac{16}{x},\ x \neq 0$.

The graph of $g$ is transformed into the graph of $f$ using two transformations. The first is a translation with vector $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$. Give a full geometric description of the second transformation. *[2 marks]*

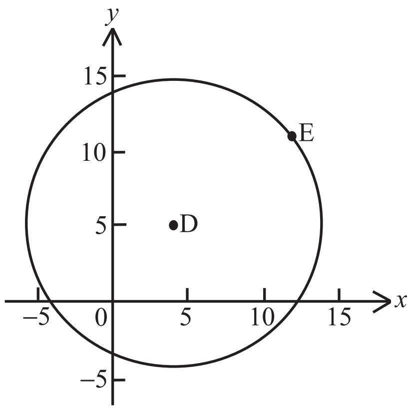
**4.** *[Maximum mark: 20]*

(i) Consider the point D with coordinates (4, 5), and the point E, with coordinates (12, 11).

(a) Find $\overrightarrow{DE}$. *[2 marks]*

(b) Find $\left|\overrightarrow{DE}\right|$. *[2 marks]*

(c) The point D is the centre of a circle and E is on the circumference as shown in the following diagram.




The point G is also on the circumference. $\overrightarrow{DE}$ is perpendicular to $\overrightarrow{DG}$. Find the possible coordinates of G. *[8 marks]*

(ii) Car 1 moves in a straight line, starting at point $A(0, 12)$. Its position $p$ seconds after it starts is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} + p\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

(a) Find the position vector of the car after 2 seconds. *[2 marks]*

Car 2 moves in a straight line starting at point $B(14, 0)$. Its position $q$ seconds after it starts is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix} + q\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Cars 1 and 2 collide at point P.

(b) (i) Find the value of $p$ and the value of $q$ when the collision occurs.

(ii) Find the coordinates of P. *[6 marks]*

**5.** *[Maximum mark: 18]*

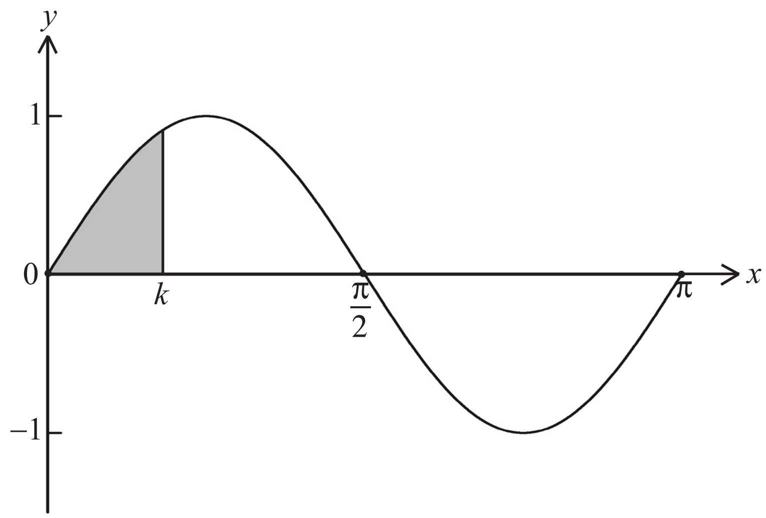
(i) A particle moves with a velocity $v$ $\text{ms}^{-1}$ given by $v = 25 - 4t^2$ where $t \geq 0$.

(a) The displacement, $s$ metres, is 10 when $t$ is 3. Find an expression for $s$ in terms of $t$. *[6 marks]*

(b) Find $t$ when $s$ reaches its maximum value. *[3 marks]*

(c) The particle has a positive displacement for $m \leq t \leq n$. Find the value of $m$ and the value of $n$. *[3 marks]*

(ii) The graph of $y = \sin 2x$ from $0 \leq x \leq \pi$ is shown below.




The area of the shaded region is 0.85. Find the value of $k$. *[6 marks]*

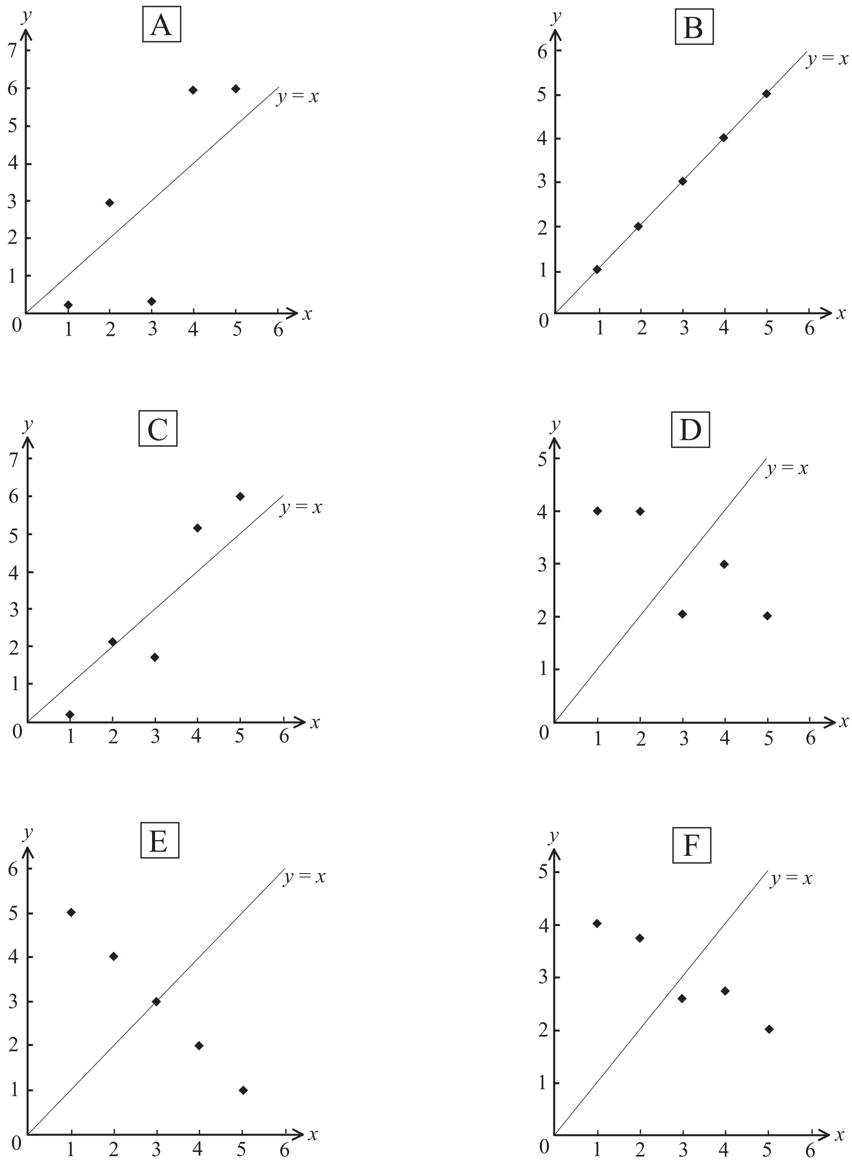
## SECTION B

*Answer **one** question from this section.*

**Statistical Methods**

**6.** *[Maximum mark: 30]*

(i) The following scatter diagrams represent six sets of data. The line $y = x$ is shown on each diagram.




The least squares regression line $y = ax + b$ is calculated for each set of data. The coefficient of linear correlation, $r$, is also calculated in each case. The values of $a$ are $-1, -0.5, 1$ or $1.5$. The values of $r$ are $\pm 0.8, \pm 0.95$ or $\pm 1$.

*(This question continues on the following page)*

*(Question 6 (i) continued)*

Write down the letter of the diagram corresponding to the following results.

(a) $a = -1,\ r = -1$; *[1 mark]*

(b) $a = 1.5,\ r = 0.95$; *[1 mark]*

(c) $a = -0.5,\ r = -0.95$; *[1 mark]*

(d) $a = 1.5,\ r = 0.8$; *[1 mark]*

(e) $a = -0.5,\ r = -0.8$. *[1 mark]*

(ii) The heights, $H$, of the people in a certain town are normally distributed with mean 170 cm and standard deviation 20 cm.

(a) A person is selected at random. Find the probability that his height is less than 185 cm. *[3 marks]*

(b) Given that $\mathrm{P}(H > d) = 0.6808$, find the value of $d$. *[3 marks]*

(iii) Three football teams, *Atletico*, *Boca* and *Celtic* play in a league. Each team plays the same number of home and away matches. The following table shows the number of goals scored in home and away matches for each team.

| | *Atletico* | *Boca* | *Celtic* |
|---|---|---|---|
| Home Goals | 45 | 30 | 20 |
| Away Goals | 20 | 25 | 20 |

It is believed that the ratio of home goals to away goals is independent of the team. This hypothesis is to be tested.

(a) Using probability theory, show that *Atletico* would be expected to score 38.6 home goals. *[2 marks]*

The following table shows the expected number of goals for each team.

| | *Atletico* | *Boca* | *Celtic* |
|---|---|---|---|
| Home Goals | 38.6 | 32.7 | 23.8 |
| Away Goals | 26.4 | 22.3 | 16.2 |

(b) Calculate the $\chi^2$ value for this data. *[2 marks]*

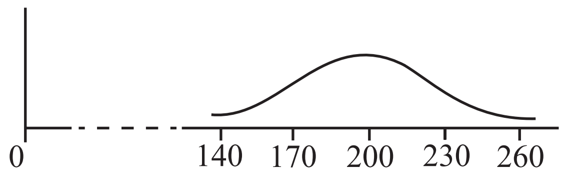
(c) (i) Is the hypothesis accepted at a 5 % level of significance?

(ii) Explain your answer to (c) (i). *[3 marks]*

*(This question continues on the following page)*

*(Question 6 continued)*

(iv) The sketch below shows the probability distribution for a normally distributed random variable, $X$.




The middle 95 % of the population lies between 140 and 260.

(a) Give a **simple** explanation why the standard deviation of $X$ would be approximately 30. *[2 marks]*

For the remainder of the question, the standard deviation of $X$ may be assumed to be 30.

(b) Samples of size $n$ are drawn from this population.

(i) Copy the above sketch of the distribution of $X$. On your drawing, sketch the distribution you would expect for the means of these samples when $n = 9$.

(ii) It is found that the middle 95 % of the **means** of the samples lie between 190 and 210. Calculate the value of $n$. *[6 marks]*

(c) A second population with the same standard deviation is believed to have a mean greater than 200. A sample of size 25 of the second population is taken and is found to have a mean of 212. Explain why, at the 5 % level of significance, this result supports the hypothesis that the mean has increased. *[4 marks]*

**Further Calculus**

**7.** *[Maximum mark: 30]*

(i) Consider the graph of the function, $f$, defined by

$$f(x) = 3x^4 - 4x^3 - 30x^2 - 36x + 112, \; -2 \le x \le 4.5 .$$

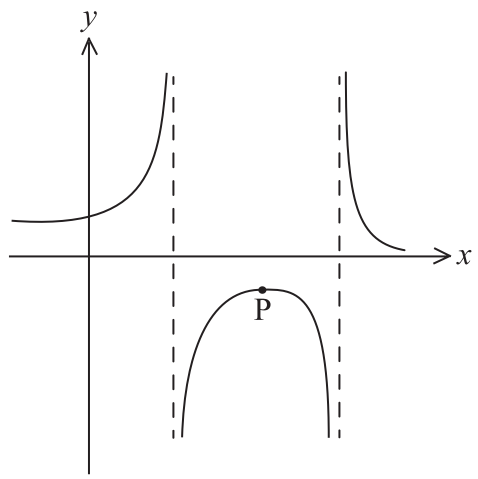
(a) Given that $f(x) = 0$ has one solution at $x = 4$, find the other solution. *[2 marks]*

(b) The tangent to the graph of $f$ is horizontal at $x = 3$ and at one other value of $x$. Find this other value. *[3 marks]*

(c) Find the $x$-coordinates of both points of inflexion on the graph of $f$. *[4 marks]*

(d) Write down **both** coordinates of the point of inflexion on the graph of $f$ where the tangent is horizontal. *[2 marks]*

A sketch of the graph of $\dfrac{1}{f}$ is given below.




(e) Write down the **equations** of the two vertical asymptotes. *[2 marks]*

(f) The tangent to the graph of $\dfrac{1}{f}$ is horizontal at P. Write down the $x$-coordinate of P. *[2 marks]*

*(This question continues on the following page)*

*(Question 7 continued)*

(ii) The function, $g$, is defined by $g(x) = \dfrac{2^x}{x^3} - 1$.

(a) Show that $g'$ is given by

$$g'(x) = \frac{2^x(x\ln 2 - 3)}{x^4}.$$ *[4 marks]*

A first approximation to the solution of $g(x) = 0$ is $x = 10$.

(b) (i) Calculate the value of $p = 10 - \dfrac{g(10)}{g'(10)}$.

(ii) Correct to three significant figures, $p$ is the solution of the equation $g(x) = 0$. Write down, **in terms of *p***, a more accurate approximation to the solution of this equation. *[3 marks]*

(iii) Consider the definite integral $I = \int_0^4 \sqrt{x^3 + 36}\,\mathrm{d}x$.
Numerical integration with a graphic display calculator gives $I = 28.5$.

(a) Use the trapezium rule with two sub-intervals to **show that** $16 + 4\sqrt{11}$ is an approximate value for $I$. *[4 marks]*

(b) (i) Does the trapezium rule overestimate or underestimate the value of $I$ in this case?

(ii) **Sketch** a diagram illustrating the above use of the trapezium rule **and** your answer to (b) (i). *[4 marks]*

**Further Geometry**

**8.** *[Maximum mark: 30]*

(i) Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are defined by

$$\boldsymbol{A} = \begin{pmatrix} 5 & 1 \\ 7 & 2 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 2 & 4 \\ -3 & 15 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 9 & -7 \\ 8 & 2 \end{pmatrix}.$$

Let $\boldsymbol{X}$ be an unknown $2 \times 2$ matrix satisfying the equation

$$\boldsymbol{AX} + \boldsymbol{B} = \boldsymbol{C}.$$

This equation may be solved for $\boldsymbol{X}$ by rewriting it in the form

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{D}$$

where $\boldsymbol{D}$ is a $2 \times 2$ matrix.

(a) Write down $\boldsymbol{A}^{-1}$. *[2 marks]*

(b) Find $\boldsymbol{D}$. *[3 marks]*

(c) Find $\boldsymbol{X}$. *[2 marks]*

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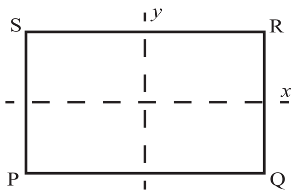
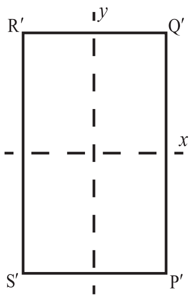
*(Question 8 continued)*

(ii) Consider the rectangle PQRS

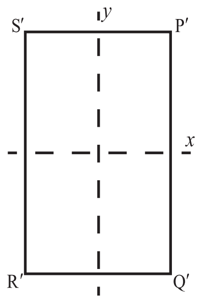



The following diagrams represent the image of the rectangle PQRS after various transformations, labelled A, B, C, D, E, F, G, H, J, K.

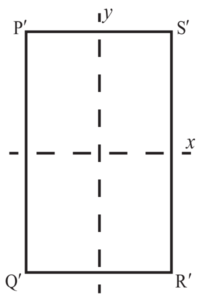
A



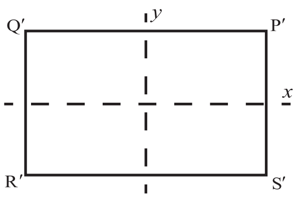

B



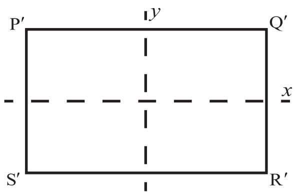

C



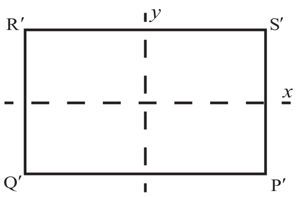

D



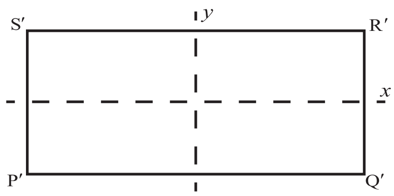

E



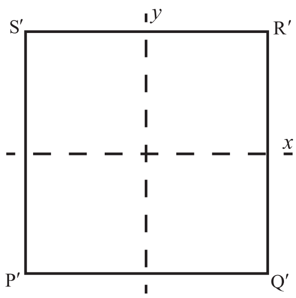

F



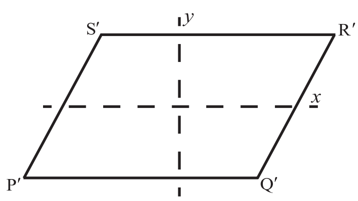

G



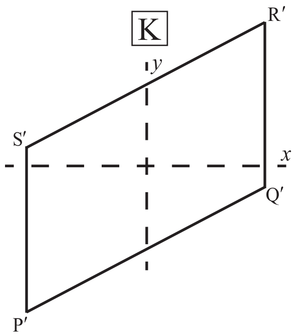

H




J




K

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*(Question 8 (ii) continued)*

The following matrices represent some of the transformations of PQRS.

$$\boldsymbol{M}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\boldsymbol{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{M}_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\boldsymbol{M}_4 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

$$\boldsymbol{M}_5 = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Copy and complete the following table matching matrices and transformations:

| Matrix | $\boldsymbol{M}_1$ | $\boldsymbol{M}_2$ | $\boldsymbol{M}_3$ | $\boldsymbol{M}_4$ | $\boldsymbol{M}_5$ |
|---|---|---|---|---|---|
| Transformation | | | | | |

*[5 marks]*

(iii) The rotation, $\boldsymbol{V}$, about the point E(3, 1) may be represented by

$$\boldsymbol{V} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}$$

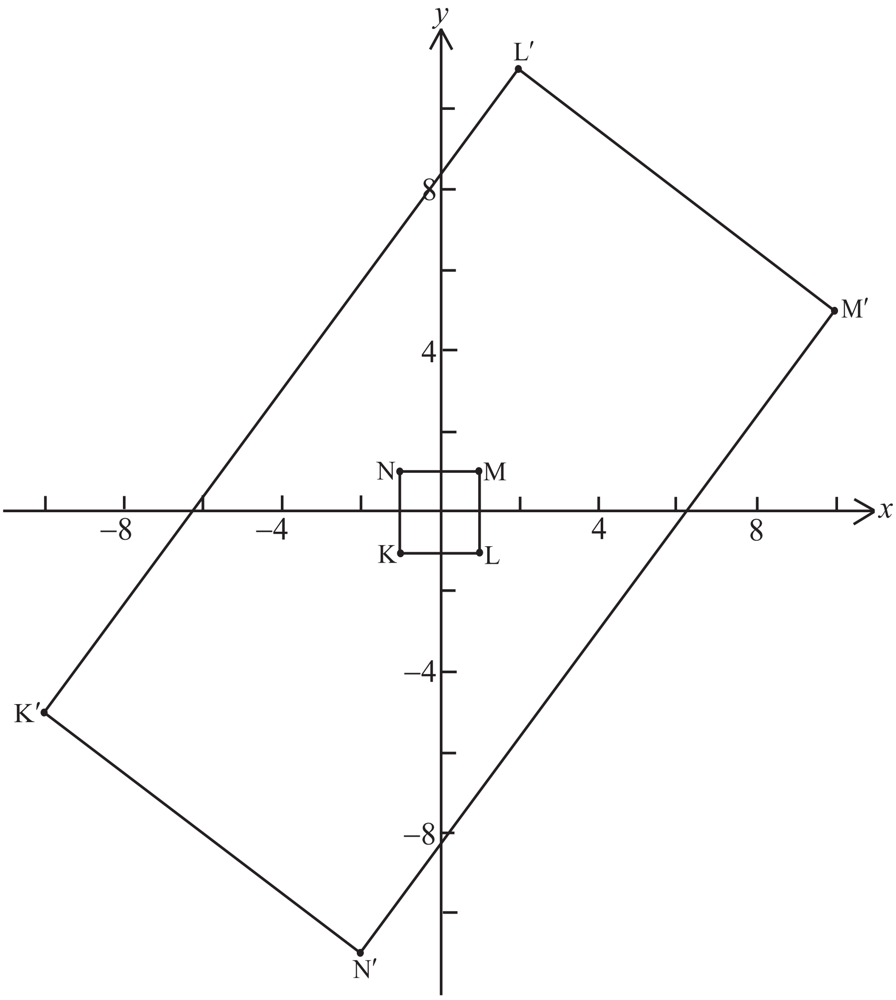
Find $c$ and $d$. *[4 marks]*

*(This question continues on the following page)*

*(Question 8 continued)*

(iv) The rectangle $K'L'M'N'$ is the image of the square KLMN under the transformation ***T***. The vertices of the square are $K(-1,-1)$, $L(1,-1)$, $M(1,1)$ and $N(-1,1)$. The vertices of the rectangle are $K'(-10,-5)$, $L'(2,11)$, $M'(10,5)$ and $N'(-2,-11)$.

The transformation is represented in the diagram below.




(a) The area of KLMN is 4. Calculate the area of $K'L'M'N'$. *[4 marks]*

The transformation ***T*** is the composite of a one-way stretch, an enlargement and one other elementary transformation.

(b) Explain why the other transformation is a reflection. *[2 marks]*

*(This question continues on the following page)*

*(Question 8 (iv) continued)*

It is given that $\boldsymbol{T} = \begin{pmatrix} 6 & 4 \\ 8 & q \end{pmatrix}$.

(c) Using part (a)

(i) calculate the area scale factor for $\boldsymbol{T}$ ;

(ii) **hence** show that $q = -3$. *[3 marks]*

The transformation $\boldsymbol{T}$ consists of the stretch $\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ followed by the reflection $\boldsymbol{R}$, followed by the enlargement $\boldsymbol{Q} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

(d) (i) Express $\boldsymbol{T}$ in terms of $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$.

(ii) Find the matrix $\boldsymbol{R}$. *[5 marks]*