IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

M05/5/MATSD/SP2/ENG/TZ0/XX+

22057402

MATHEMATICAL STUDIES
STANDARD LEVEL
PAPER 2

Wednesday 4 May 2005 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 12]*

A small manufacturing company makes and sells x machines each month. The monthly cost C, in dollars, of making x machines is given by

$$C(x) = 2600 + 0.4x^2 .$$

The monthly income I, in dollars, obtained by selling x machines is given by

$$I(x) = 150x - 0.6x^2 .$$

(a) Show that the company's monthly profit can be calculated using the quadratic function

$$P(x) = -x^2 + 150x - 2600 .$$ *[2 marks]*

(b) The maximum profit occurs at the vertex of the function $P(x)$. How many machines should be made and sold each month for a maximum profit? *[2 marks]*

(c) If the company does maximize profit, what is the selling price of each machine? *[4 marks]*

(d) Given that $P(x) = (x-20)(130-x)$, find the smallest number of machines the company must make and sell each month in order to make **positive** profit. *[4 marks]*

2. *[Maximum mark: 12]*

(i) (a) A farmer wants to construct a new fence across a field. The plan is shown below. The new fence is indicated by a dotted line.

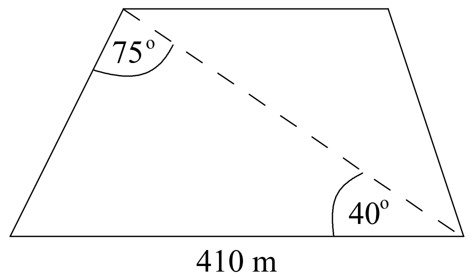


Diagram not to scale

Calculate the length of the fence. *[5 marks]*

(b) The fence creates two sections of land. Find the area of the smaller section of land ABC, given the additional information shown below.

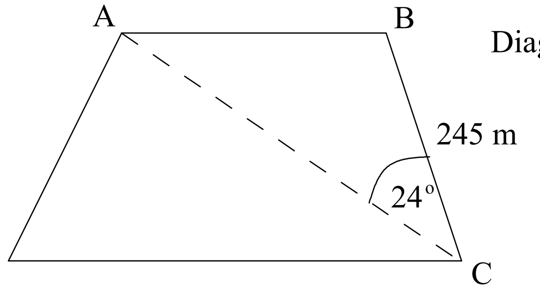


Diagram not to scale

[3 marks]

(ii) Find the volume of the following prism.

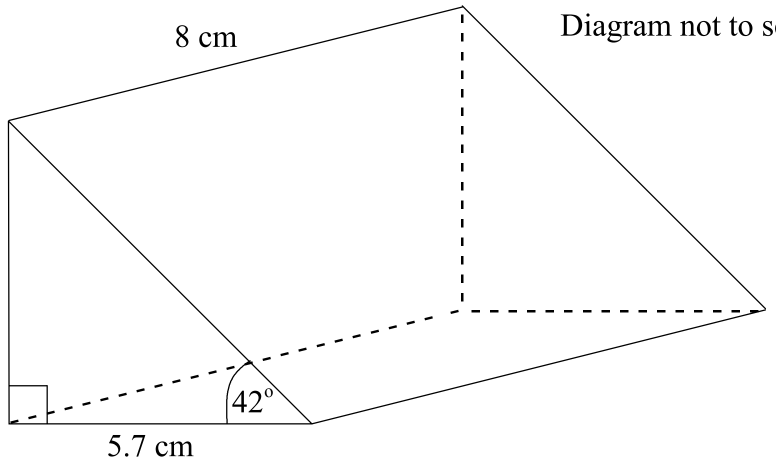


Diagram not to scale

[4 marks]

3. *[Maximum mark: 17]*

(i) Let p stand for the proposition "I will walk to school". Let q stand for the proposition "the sun is shining".

(a) Write the following statements in symbolic logic form

(i) "If the sun is shining then I will walk to school."

(ii) "If I do not walk to school then the sun is not shining." *[4 marks]*

(b) Write down, in words, the converse of the statement

"If the sun is shining then I will walk to school." *[2 marks]*

(ii) (a) **Copy** and **complete** the table below by filling in the three empty columns.

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$(p \vee q) \wedge \neg p \Rightarrow q$
T	T	T	T			
T	F	F	T			
F	T	F	T			
F	F	F	F			

[3 marks]

(b) What word is used to describe the argument $(p \vee q) \wedge \neg p \Rightarrow q$? *[1 mark]*

(iii) The following results were obtained from a survey concerning the reading habits of students.

60 % read magazine P
50 % read magazine Q
50 % read magazine R
30 % read magazines P and Q
20 % read magazines Q and R
30 % read magazines P and R
10 % read all three magazines

(a) Represent all of this information on a Venn diagram. *[4 marks]*

(b) What percentage of students read exactly two magazines? *[1 mark]*

(c) What percentage of students read at least two magazines? *[1 mark]*

(d) What percentage of students do not read any of the magazines? *[1 mark]*

4. *[Maximum mark: 14]*

The data in the table below refers to a sample of 60 randomly chosen plants.

Growth rate ↓	Classification by environment			
	dark	light	shady	total
high	3	8	14	25
low	8	9	18	35
total	11	17	32	60

(a) (i) Find the probability of a plant being in a shady environment.

(ii) Find the probability of a plant having a low growth rate and being in a dark environment.

(iii) Find the probability of a plant **not** being in a dark environment. *[5 marks]*

(b) A plant is chosen at random from the above group.

Find the probability that the chosen plant has

(i) a high growth rate or is in a dark environment, but not both

(ii) a light environment, given that it has a high growth rate. *[4 marks]*

(c) The 60 plants in the above group were then classified according to leaf type. It was found that 15 of the plants had type A leaves, 37 had type B leaves and 8 had type C leaves.

Two plants were randomly selected from this group. Find the probability that

(i) both plants had type C leaves

(ii) neither of the plants had type B leaves. *[5 marks]*

5. *[Maximum mark: 15]*

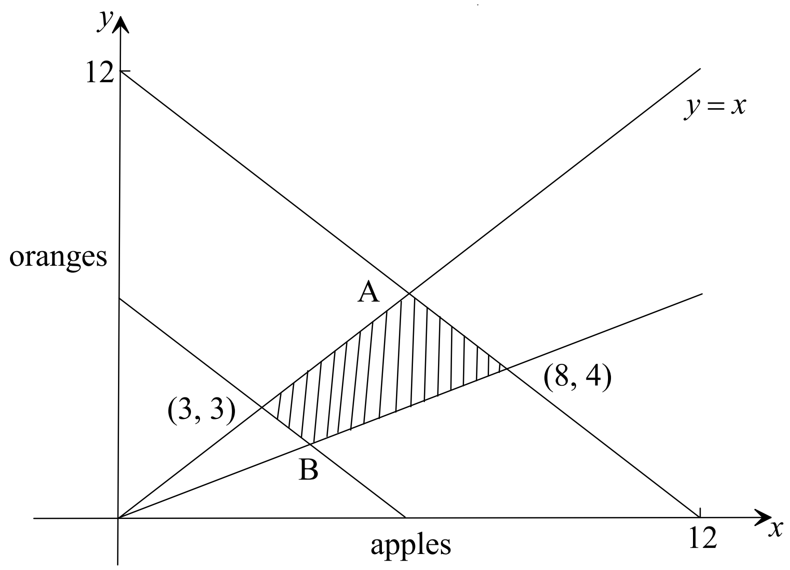
Cedric's mother has asked him to buy some fruit.
Cedric must obey the following instructions.

He must buy at least as many apples as oranges.
He must buy at least 6 pieces of fruit.
He must **not** buy more than 12 pieces of fruit.
The number of apples bought should not be more than twice the number of oranges bought.

Taking x to be the number of apples and y the number of oranges, the first condition becomes $x \geq y$.

(a) Write down **three more** inequalities expressing the remaining conditions set by Cedric's mother. *[3 marks]*

In the diagram below, the four boundary lines for Cedric's four inequalities are drawn. The equation for one of the lines is given as well as the coordinates of two points of intersection and two axis crossings.



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(Question 5 continued)

(b) (i) Copy the diagram and label the three remaining boundary lines with their equations from part (a). (Graph paper is not required for this.)

(ii) Calculate the coordinates of the points A and B. *[7 marks]*

Apples cost \$ 2 and oranges cost \$ 1 each. Cedric's mother gives him \$ 11. As long as he follows all her instructions, he can keep any change.

(c) (i) How many of each type of fruit should Cedric buy in order to keep as much change as possible for himself? Briefly explain your reasoning.

(ii) If Cedric buys the numbers of fruits in part (c)(i) calculate how much change he keeps for himself? *[5 marks]*

SECTION B

*Answer **one** question from this section.*

Matrices and Graph Theory

6. *[Maximum mark: 30]*

(i) Let $\boldsymbol{A}$ be the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -3 \\ c & 3 \end{pmatrix}$ where c is a constant.

(a) Write down the transpose $\boldsymbol{A}^{\mathrm{T}}$ of $\boldsymbol{A}$. *[1 mark]*

(b) (i) Calculate $\det(\boldsymbol{A})$ the determinant of $\boldsymbol{A}$.

(ii) Find the value of c for which $\boldsymbol{A}$ is a singular matrix. *[4 marks]*

(c) When $c = 1$ show that $\boldsymbol{A}^2 - 5\boldsymbol{A} = -\det(\boldsymbol{A}) \times \boldsymbol{I}$ for $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *[7 marks]*

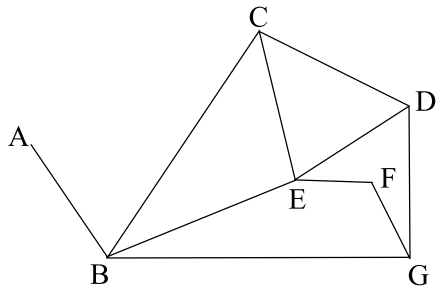
(This question continues on the following page)

(Question 6 continued)

(ii) (a) (i) Draw a connected graph with four vertices labelled A, B, C and D.

(ii) Draw a disconnected graph with four vertices labelled A, B, C and D. *[4 marks]*

(b) The graph shown below represents power lines connecting seven towns A to G.



(i) Write down the degree of the vertex C in the graph.

The adjacency matrix for the graph is

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D & E & F & G \end{matrix} \\ \boldsymbol{M} = \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{pmatrix} 0 & 1 & x & 0 & 0 & 0 & 0 \\ 1 & 0 & y & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(ii) Find the values of x and y in this matrix.

The square of the matrix $\boldsymbol{M}$ is

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E & F & G \end{matrix} \\ \boldsymbol{M}^2 = \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 4 & 1 & 3 & 1 & 2 & 0 \\ 1 & 1 & 3 & 1 & 2 & 1 & 2 \\ 0 & 3 & 1 & 3 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 & 4 & 0 & 3 \\ 0 & 2 & 1 & 2 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 3 & 0 & k \end{pmatrix} \end{array}$$

(iii) How many different routes are there from town E to town G requiring exactly two steps?

(iv) Find the value of k. *[6 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) Yuichi and Katerina are playing a two person zero sum game. On each turn, both players must choose one of three animals. The payoff matrix for Yuichi is shown below.

		Katerina		
		Lion	Mouse	Dog
Yuichi	Elephant	1	–3	2
	Wolf	–2	1	3
	Cat	–4	4	–3

(a) What is the result for Yuichi if he chooses elephant and Katerina chooses dog? *[1 mark]*

(b) Which animals should each of Yuichi and Katerina choose in order to adopt a play safe strategy? *[2 marks]*

(c) Suppose that Yuichi guesses that Katerina will play safe.

(i) Which animal should Yuichi choose now?

(ii) What is the result if he makes this choice? *[2 marks]*

(d) Katerina is very smart and guesses that Yuichi will make the choice in part (c).

(i) Which animal is now the best choice for Katerina?

(ii) How much will she win with the choice from (d)(i)? *[2 marks]*

(e) Is the play safe strategy for this game stable or unstable? *[1 mark]*

Further Statistics and Probability

7. *[Maximum mark: 30]*

(i) A variable X follows a normal distribution with a mean of –5 and standard deviation of 9.

Find the probability that a randomly chosen item from this population has a negative value. *[2 marks]*

(ii) A firm delivers milk to a number of homes in a street. The time taken to complete these deliveries follows a normal distribution with mean of 10 minutes and standard deviation 1.5 minutes. Milk is delivered every day of the week.

(a) What percentage of the days of the year would you expect the milk deliveries to take 10 minutes or less? *[1 mark]*

(b) (i) On a suitable diagram, show the position of the mean time taken, the position of one positive standard deviation from the mean, and the 12 minute mark.

(ii) On your diagram, shade the area under the curve that represents the number of days that the milk deliveries take more than 12 minutes.

(iii) In a period of 200 days, calculate the number of days when the milk deliveries take between 11.5 and 12 minutes. *[6 marks]*

(c) Find how many minutes are needed for the fastest 5 % of deliveries. *[2 marks]*

(This question continues on the following page)

(Question 7 continued)

(iii) Workers in a particular industry were trained to use a more efficient method in their work. In one firm, one hundred workers were trained. Their productivity was measured before and after training. The firm's results were then compared with the results for the whole industry.

Improvement x %	Frequency Firm's data	Expected frequency Industry data	$f_e - f_o$	$\frac{(f_e - f_o)^2}{f_e}$
$-10 < x \le 5$	10	15.9	5.9	2.189
$5 < x \le 10$	48	44.0	–4.0	0.364
$10 < x \le 15$	39	p		
$15 < x \le 25$	3	6.7	3.7	2.043
	[100]			

The firm's manager believes that the change in the employees' productivity is the same as that for the whole industry. The industry data can be modelled by a **normal distribution** with a mean of 9 % and standard deviation of 4 %.

(a) State suitable null and alternate hypotheses for a χ^2 'goodness of fit' test of the employees' data. *[2 marks]*

(b) Write down the value of p, in the expected frequency column. *[1 mark]*

(c) Calculate the χ^2 statistic for this data. *[3 marks]*

(d) Write down the number of degrees of freedom v. *[1 mark]*

(e) Carry out a χ^2 'goodness of fit' analysis to test your hypothesis at the 5 % level. Clearly state your conclusion. *[3 marks]*

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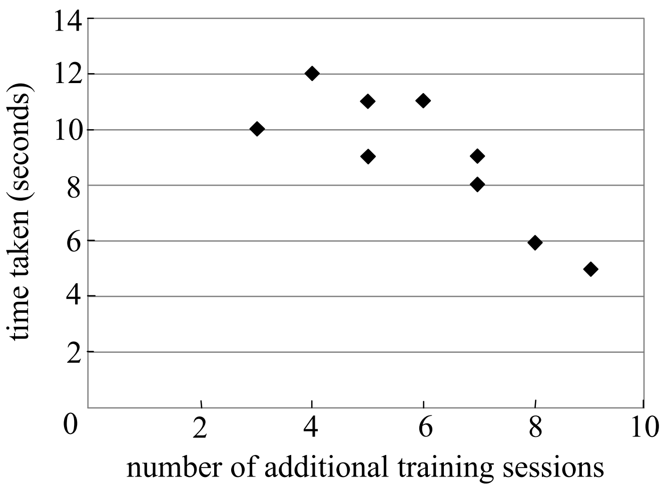
(Question 7 continued)

(iv) A number of employees at a different factory were given x additional training sessions each. They were then timed on how long (y seconds) it took them to complete a task. The results are shown in the scatter diagram below. A list of descriptive statistics is also given.



$n = 9$,
sum of x values: $\sum x = 54$,
sum of y values: $\sum y = 81$,
mean of x values: $\bar{x} = 6$,
mean of y values: $\bar{y} = 9$,
standard deviation of x: $s_x = 1.94$,
standard deviation of y: $s_y = 2.35$,
covariance: $s_{xy} = -3.77$.

(a) Determine the product-moment correlation coefficient (r) for this data. *[2 marks]*

(b) What is the nature of the relationship between the amount of additional training and the time taken to complete the task? *[2 marks]*

(c) Calculate $\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$ given that the covariance $s_{xy} = -3.77$. *[1 mark]*

(d) (i) Determine the equation of the linear regression line for y on x.

(ii) Find the expected time to complete the task for an employee who only attended three additional training sessions. *[4 marks]*

Introductory Differential Calculus

8. *[Maximum mark: 30]*

(i) The function $f(x)$ is given by the formula

$$f(x) = 2x^3 - 5x^2 + 7x - 1$$

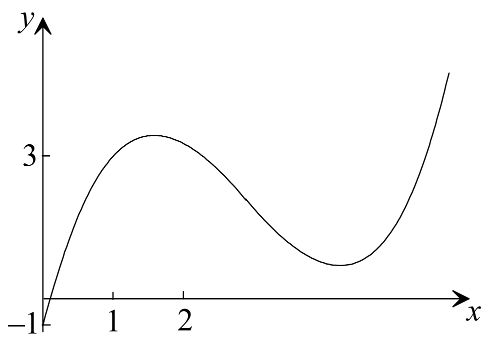
(a) Evaluate $f(1)$. *[2 marks]*

(b) Calculate $f'(x)$. *[3 marks]*

(c) Evaluate $f'(2)$. *[2 marks]*

(d) State whether the function $f(x)$ is increasing or decreasing at $x = 2$. *[1 mark]*

(e) The sketch graph shown below is the graph of a cubic function.



(i) Is it possible that this is the graph of the function $f(x)$ above?

(ii) State one reason for your decision. *[2 marks]*

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(Question 8 continued)

(ii) Consider the expression $\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\left(\frac{2(x+h)^2 - 3(x+h) - 2x^2 + 3x}{h}\right)$

(a) (i) Expand each of the brackets in the **numerator** of the limit above.

(ii) Use the result obtained in (a)(i) to simplify the whole of the numerator in the limit.

(iii) Evaluate the limit and write down an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. *[5 marks]*

(b) Write down an expression for a function $y(x)$ whose derivative is the $\frac{\mathrm{d}y}{\mathrm{d}x}$ given by the limit in part (a). *[2 marks]*

(iii) A ball is thrown upwards so that its height h, in metres, depends on time t, in seconds, as follows:

$$h(t) = 3 + 16t - 4.9t^2$$

(a) Calculate the height of the ball at the moment it was thrown. *[2 marks]*

(b) (i) Find an expression for the velocity of the ball $v(t)$.

(ii) Evaluate the velocity of the ball at the moment it was thrown. *[4 marks]*

(c) (i) Show that the ball reaches its maximum height at approximately $t = 1.63$ seconds.

(ii) Calculate the maximum height reached. *[4 marks]*

(d) (i) Find an expression for the acceleration of the ball.

(ii) State the direction of the acceleration. *[3 marks]*