N04/5/MATME/SP2/ENG/TZ0/XX

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MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Thursday 4 November 2004 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Indicate the make and model of your calculator in the appropriate box on your cover sheet.

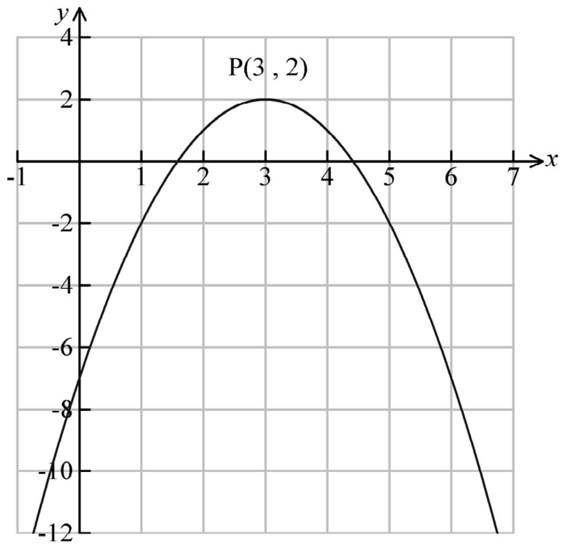
Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 13]*

The function $f(x)$ is defined as $f(x) = -(x-h)^2 + k$. The diagram below shows part of the graph of $f(x)$. The maximum point on the curve is $\mathrm{P}(3, 2)$.



(a) Write down the value of

(i) h;

(ii) k. *[2 marks]*

(b) Show that $f(x)$ can be written as $f(x) = -x^2 + 6x - 7$. *[1 mark]*

(c) Find $f'(x)$. *[2 marks]*

The point Q lies on the curve and has coordinates $(4, 1)$. A straight line L, through Q, is perpendicular to the tangent at Q.

(d) (i) Calculate the gradient of L.

(ii) Find the equation of L.

(iii) The line L intersects the curve again at R. Find the x-coordinate of R. *[8 marks]*

2. *[Maximum mark: 15]*

Points A, B, and C have position vectors $4\boldsymbol{i}+2\boldsymbol{j}$, $\boldsymbol{i}-3\boldsymbol{j}$ and $-5\boldsymbol{i}-5\boldsymbol{j}$. Let D be a point on the x-axis such that ABCD forms a parallelogram.

(a) (i) Find $\overrightarrow{\mathrm{BC}}$.

(ii) Find the position vector of D. *[4 marks]*

(b) Find the angle between $\overrightarrow{\mathrm{BD}}$ and $\overrightarrow{\mathrm{AC}}$. *[6 marks]*

The line L_1 passes through A and is parallel to $\boldsymbol{i}+4\boldsymbol{j}$. The line L_2 passes through B and is parallel to $2\boldsymbol{i}+7\boldsymbol{j}$. A vector equation of L_1 is $\boldsymbol{r}=(4\boldsymbol{i}+2\boldsymbol{j})+s(\boldsymbol{i}+4\boldsymbol{j})$.

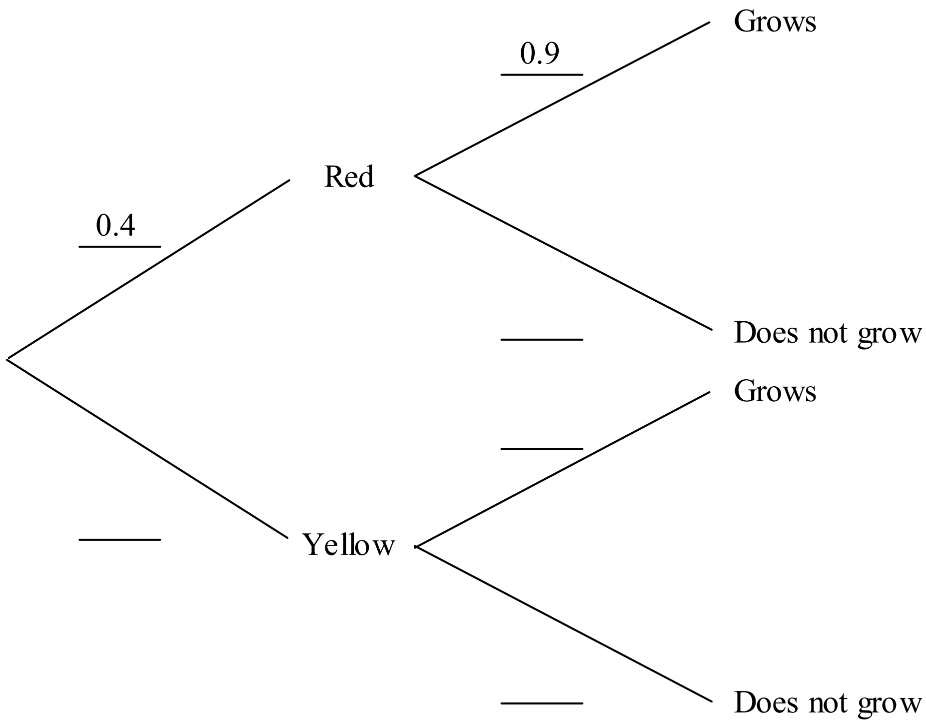
(c) Write down a vector equation of L_2 in the form $\boldsymbol{r}=\boldsymbol{b}+t\boldsymbol{q}$. *[1 mark]*

(d) The lines L_1 and L_2 intersect at the point P. Find the position vector of P. *[4 marks]*

3. *[Maximum mark: 10]*

A packet of seeds contains 40 % red seeds and 60 % yellow seeds. The probability that a red seed grows is 0.9, and that a yellow seed grows is 0.8. A seed is chosen at random from the packet.

(a) On your **answer sheet,** copy and complete the probability tree diagram below.



[3 marks]

(b) (i) Calculate the probability that the chosen seed is red and grows.

(ii) Calculate the probability that the chosen seed grows.

(iii) Given that the seed grows, calculate the probability that it is red. *[7 marks]*

4. *[Maximum mark: 11]*

A company offers its employees a choice of two salary schemes A and B over a period of 10 years.

Scheme A offers a starting salary of \$11 000 in the first year and then an annual increase of \$400 per year.

(a) (i) Write down the salary paid in the second year and in the third year.

(ii) Calculate the **total** (amount of) salary paid over ten years. *[3 marks]*

Scheme B offers a starting salary of \$10 000 dollars in the first year and then an annual increase of 7 % of the previous year's salary.

(b) (i) Write down the salary paid in the second year and in the third year.

(ii) Calculate the salary paid in the tenth year. *[4 marks]*

(c) Arturo works for n complete years under scheme A. Bill works for n complete years under scheme B. Find the minimum number of years so that the total earned by Bill exceeds the total earned by Arturo. *[4 marks]*

5. *[Maximum mark: 21]*

(i) Let $f(x) = 1 + 3\cos(2x)$ for $0 \le x \le \pi$, and x is in radians.

(a) (i) Find $f'(x)$.

(ii) Find the values for x for which $f'(x) = 0$, giving your answers in terms of π. *[6 marks]*

The function $g(x)$ is defined as $g(x) = f(2x) - 1,\ 0 \le x \le \frac{\pi}{2}$.

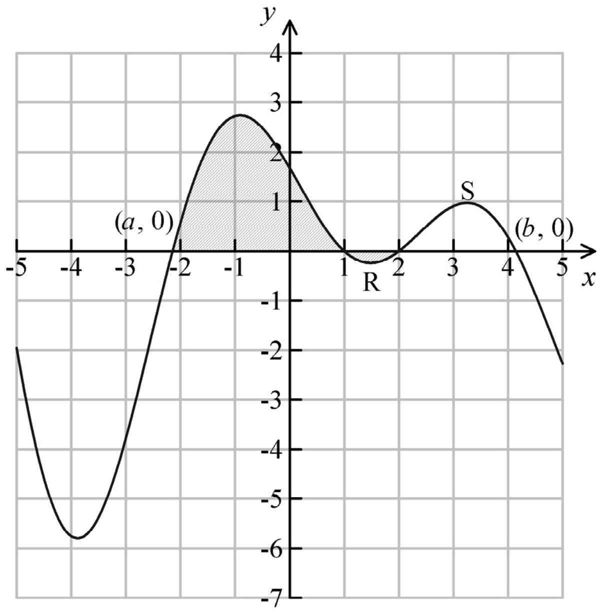
(b) (i) The graph of f may be transformed to the graph of g by a stretch in the x-direction with scale factor $\frac{1}{2}$ followed by another transformation. Describe fully this other transformation.

(ii) Find the solution to the equation $g(x) = f(x)$. *[4 marks]*

(This question continues on the following page)

(Question 5 continued)

(ii) Let $h(x) = (x-2)\sin(x-1)$ for $-5 \le x \le 5$. The curve of $h(x)$ is shown below. There is a minimum point at R and a maximum point at S. The curve intersects the x-axis at the points $(a, 0)$ $(1, 0)$ $(2, 0)$ and $(b, 0)$.



(a) Find the exact value of

(i) a;

(ii) b. *[2 marks]*

The regions between the curve and the x-axis are shaded for $a \le x \le 2$ as shown.

(b) (i) Write down an expression which represents the **total** area of the shaded regions.

(ii) Calculate this total area. *[5 marks]*

(c) (i) The y-coordinate of R is -0.240. Find the y-coordinate of S.

(ii) Hence or otherwise, find the range of values of k for which the equation $(x-2)\sin(x-1) = k$ has **four** distinct solutions. *[4 marks]*

SECTION B

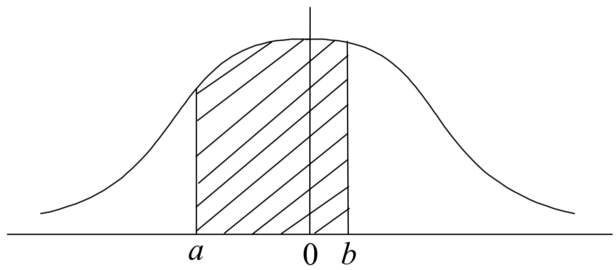
*Answer **one** question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) Reaction times of human beings are normally distributed with a mean of 0.76 seconds and a standard deviation of 0.06 seconds.

(a) The graph below is that of the **standard** normal curve. The shaded area represents the probability that the reaction time of a person chosen at random is between 0.70 and 0.79 seconds.



(i) Write down the value of a and of b.

(ii) Calculate the probability that the reaction time of a person chosen at random is

(a) greater than 0.70 seconds;

(b) between 0.70 and 0.79 seconds. *[6 marks]*

Three percent (3 %) of the population have a reaction time less than c seconds.

(b) (i) Represent this information on a diagram similar to the one above. Indicate clearly the area representing 3 %.

(ii) Find c. *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(ii) The contents of bottles of mineral water are normally distributed with a mean of μ ml and a standard deviation of 2.5 ml.

(a) Let the null hypothesis be that $\mu = 298$. Assuming that the null hypothesis is true, find the probability that a random sample of 25 bottles has a mean of 297.2 ml or less. *[3 marks]*

(b) A random sample of 25 bottles was found to have a mean of 297.2 ml.

(i) At the 5 % level of significance, which of the following hypotheses would you accept?

$$H_0: \ \mu = 298$$
$$H_1: \ \mu < 298$$

State the reason for your answer.

(ii) Calculate the 99 % confidence interval for μ. *[7 marks]*

(iii) The table below shows data on the employment situation of a random group of men and women in a city.

	Unemployed	Employed Part-Time	Employed Full-Time
Men	8	11	16
Women	20	10	10

A chi-squared test is to be carried out to determine whether the employment situation is independent of gender. The following is the contingency table for the expected frequencies based on the above data.

	Unemployed	Employed Part-Time	Employed Full-Time
Men	13.07	p	q
Women	r	s	t

(a) State the assumption under which the values in the contingency table are calculated. *[2 marks]*

(b) Calculate the value of r and of t. *[3 marks]*

(c) Calculate the value of χ^2 for the given data. *[2 marks]*

(d) At the 5 % significance level, state your conclusion regarding whether or not the employment situation is independent of gender. Give reasons for your answer. *[3 marks]*

Further Calculus

7. *[Maximum mark: 30]*

(i) Let $f(x) = \dfrac{1}{1+x^2}$.

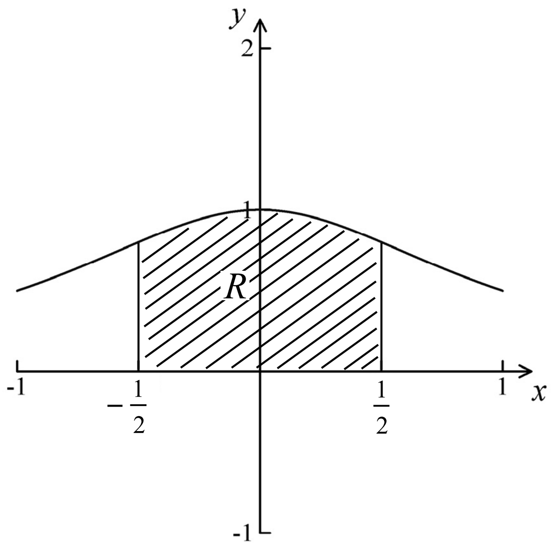
(a) Write down the equation of the horizontal asymptote of the graph of *f*. *[1 mark]*

(b) Find $f'(x)$. *[3 marks]*

(c) The second derivative is given by $f''(x) = \dfrac{6x^2 - 2}{\left(1+x^2\right)^3}$.

Let A be the point on the curve of *f* where the gradient of the tangent is a maximum. Find the *x*-coordinate of A. *[4 marks]*

(d) Let *R* be the region under the graph of *f*, between $x = -\dfrac{1}{2}$ and $x = \dfrac{1}{2}$, as shaded in the diagram below.



(i) Write down the definite integral which represents the area of *R*.

(ii) Use the trapezium rule with four sub-intervals to estimate the area of *R*.

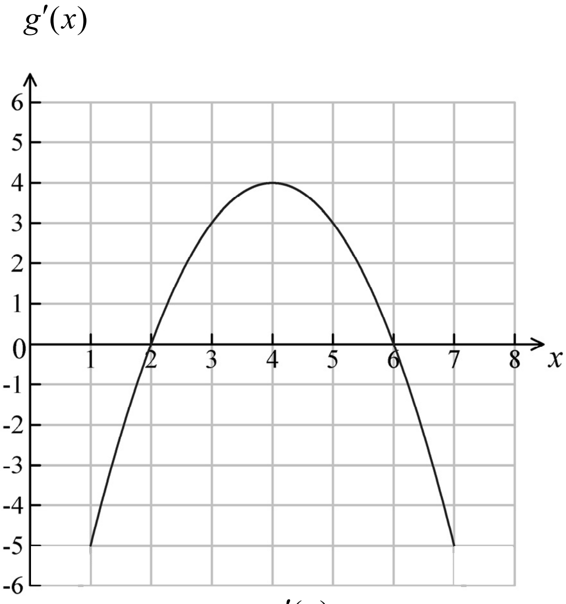
(iii) Explain briefly why the area of *R* is greater than the estimate using the trapezium rule. *[6 marks]*

(This question continues on the following page)

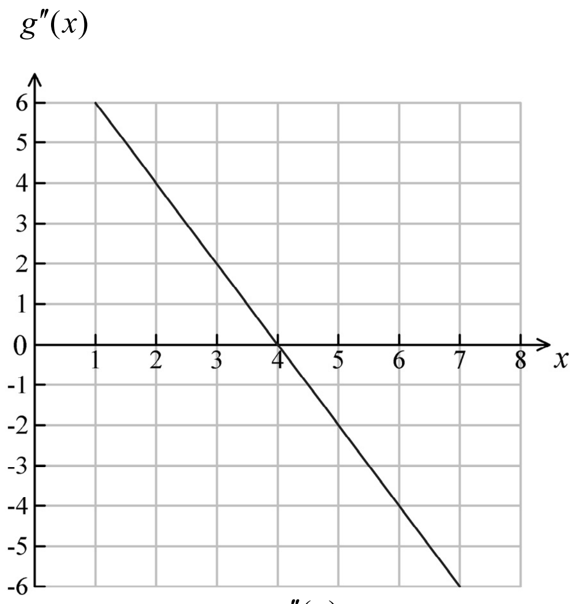
(Question 7 continued)

(ii) Let $y = g(x)$ be a function of x for $1 \le x \le 7$. The graph of g has an inflexion point at P, and a minimum point at M.

Partial sketches of the curves of g' and g'' are shown below.



$y = g'(x)$



$y = g''(x)$

Use the above information to answer the following.

(a) Write down the x-coordinate of P, **and** justify your answer. *[2 marks]*

(b) Write down the x-coordinate of M, **and** justify your answer. *[2 marks]*

(c) Given that $g(4) = 0$, sketch the graph of g. On the sketch, mark the points P and M. *[4 marks]*

(iii) Let $h(x) = \dfrac{\tan x + 1}{\cos^2 x}$.

(a) Using the substitution $u = \tan x$ or otherwise, find $\int h(x)\,\mathrm{d}x$. *[4 marks]*

(b) Show that $\int_0^{\frac{\pi}{4}} h'(x)\,\mathrm{d}x = 3$. *[4 marks]*

Further Geometry

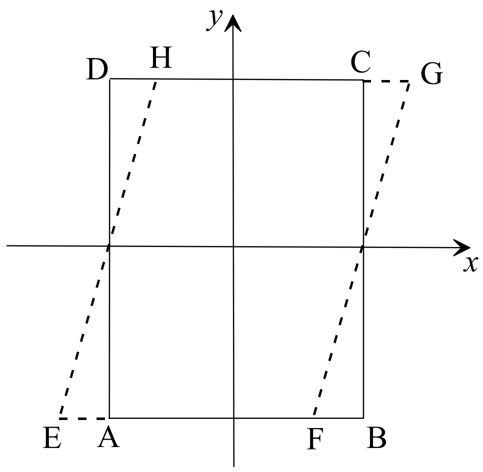
8. *[Maximum mark: 30]*

(i) (a) The transformation $\boldsymbol{S}$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

(i) Give a complete geometric description of $\boldsymbol{S}$.

(ii) What is the area scale factor for $\boldsymbol{S}$? *[4 marks]*

(b) Another transformation, $\boldsymbol{U}$, of this type, transforms the rectangle ABCD with vertices $\mathrm{A}(-3,-4), \mathrm{B}(3,-4), \mathrm{C}(3,4), \mathrm{D}(-3,4)$ into the parallelogram EFGH with vertices $\mathrm{E}(-4,-4), \mathrm{F}(2,-4), \mathrm{G}(4,4)$, $\mathrm{H}(-2,4)$. The diagram below illustrates this transformation.



Write down

(i) the equation of the invariant line of the transformation $\boldsymbol{U}$;

(ii) the scale factor for $\boldsymbol{U}$.

Find

(iii) the area of ABCD;

(iv) the area of EFGH. *[4 marks]*

(This question continues on the following page)

(Question 8 (i) continued)

The transformation ***T*** is represented by the matrix $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$. ***T*** is applied to the parallelogram EFGH, transforming E to $\mathrm{J}(-16, -16)$ and F to $\mathrm{K}(2, -4)$. L is the image of G and M is the image of H.

(c) Find the coordinates of

(i) L;

(ii) M. *[4 marks]*

(d) Calculate the area of JKLM. *[2 marks]*

(e) On graph paper, using a scale of 1 cm = 2 units, draw the figures

(i) EFGH;

(ii) JKLM. *[3 marks]*

(f) (i) From the graph, identify a vertex other than F that is its own image.

(ii) Hence, draw in the invariant line, L_1, on which every point is its own image under ***T***.

(iii) Write down the equation of L_1. *[3 marks]*

(g) From the graph, it should also be apparent that there is another line, L_2, for which the image of every point on the line is a different point (other than $(0, 0)$) on the same line. Draw this line on your graph. *[1 mark]*

(h) JKLM is transformed back into EFGH by the transformation ***V***. Write down the matrix representing ***V***. *[2 marks]*

(This question continues on the following page)

(Question 8 continued)

(ii) The line with equation $3x - 4y = 0$ makes an angle θ with the x-axis.

It can be shown that $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. The transformation $\boldsymbol{R}$ is a reflection in the line $3x - 4y = 0$.

(a) Show that the matrix representing $\boldsymbol{R}$ is $\frac{1}{25}\begin{pmatrix} 7 & 24 \\ 24 & -7 \end{pmatrix}$. *[3 marks]*

Another reflection $\boldsymbol{P}$ is represented by $\boldsymbol{P}:\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \boldsymbol{R}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -6 \\ 8 \end{pmatrix}$.

(b) Give the equation of the line of reflection for $\boldsymbol{P}$. *[3 marks]*

(c) Give a general description of other invariant lines for this transformation. *[1 mark]*